LECTURE 2: OPERATOR ALGEBRAS AND FREENESS

1. Group von Neumann algebras and free group factors

1.1. **Locally Compact Groups and Haar Measure.** Let G be a group endowed with a topology making G a locally compact topological space. This means that every point of G has a compact neighbourhood. Then there exists a regular Borel measure μ_L on G which is invariant under left translation. Invariance under left translation means that for any Borel set $B \subseteq G$ and any group element $g \in G$ we have

$$\mu_L(gB) = \mu_L(B). \tag{1}$$

Regularity means that

$$\mu_L(B) = \inf\{\mu_L(U) : U \supseteq B,\ U \text{ open}\} \tag{2}$$

$$= \sup\{\mu_L(K) : K \supseteq B,\ K \text{ compact}\}. \tag{3}$$

Such a measure μ_L is called a *left Haar measure* on G. One also has the existence of a right Haar measure μ_R which has the right translation invariance property; left and right Haar measures may or may not coincide. In the special case that they do, G is called a *unimodular* group, and we denote the left/right Haar measure simply μ.

1.2. **The Group Algebra.** Existence of Haar measure allows us to integrate over G. Denote by $C_C(G)$ the collection of complex-valued compactly supported functions on G. Define the *convolution* of two functions $a, b \in C_C(G)$ by

$$(a * b)(g) := \int_G a(h)b(h^{-1}g)d\mu(h). \tag{4}$$

One also defines an involution on $C_C(G)$ by

$$a^*(g) = \overline{a(g^{-1})}. \tag{5}$$

If G is a discrete group, i.e. $\{g\}$ is an open set for any $g \in G$, then compactly supported means finitely supported. Thus $C_C(G)$ is the collection of finitely supported functions on G, and so can be identified with the *group algebra* $\mathbb{C}[G]$ of formal linear combinations of elements in g with complex coefficients:

$$a = \sum_{g \in G} a(g)g \qquad \text{where only finitely many } \alpha(g) \neq 0. \tag{6}$$

Then multiplication is written

$$a * b = \sum_{g\in G}(a * b)(g)g = \sum_{g\in G}(\sum_{h\in G} a(h)b(h^{-1}g))g. \tag{7}$$

Note that the function

$$\delta_e = 1 \cdot e \tag{8}$$

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is the identity element in the group algebra $\mathbb{C}[G]$, where e is the identity element in G. A discrete group G is locally compact and unimodal with left/right Haar measure of each group element equal to 1.

Now define an inner product on $\mathbb{C}[G]$ by setting

$$\langle g, h\rangle = \begin{cases} 1, & \text{if } g = h \\ 0, & \text{if } g \neq h \end{cases} \tag{9}$$

on G and extending sesquilineary to $\mathbb{C}[G]$. From this inner product we define the 2-norm on $\mathbb{C}[G]$ by

$$\|a\|_2^2 = \langle a, a\rangle. \tag{10}$$

In this way $(\mathbb{C}[G], \|\cdot\|)$ is a normed $*$-algebra. However, it is not complete. The completion of $\mathbb{C}[G]$ with respect $\|\cdot\|_2$ consists of all functions $a : G \to \mathbb{C}$ satisfying

$$\sum_{g\in G} |a(g)|^2 < \infty, \tag{11}$$

and is denoted by $\ell_2(G)$, and is a Hilbert space.

Now consider a unitary group representation

$$\lambda : G \to \mathcal{U}(\ell_2(G)) \tag{12}$$

defined by

$$\lambda(g) \cdot \sum_{h\in G} a(h)h := \sum_{h\in G} a(h)(gh). \tag{13}$$

This is the left regular representation of G on the Hilbert space $\ell_2(G)$. It is obvious from the definition that each $\lambda(g)$ is an isometry of $\ell_2(G)$, but we want to check that it is in fact a unitary operator on $\ell_2(G)$. Since clearly

$$\langle gh, k\rangle = \langle h, g^{-1}k\rangle, \tag{14}$$

the adjoint of the operator $\lambda(g)$ is $\lambda(g^{-1})$. But then since λ is a group homomorphism, we have $\lambda(g)\lambda(g)^* = I = \lambda(g)^*\lambda(g)$, so that $\lambda(g)$ is indeed a unitary operator on $\ell_2(G)$.

Now extend the domain of λ from G to $\mathbb{C}[G]$ in the obvious way:

$$\lambda(a) = \lambda(\sum_{g\in G} a(g)g) = \sum_{g\in G} a(g)\lambda(g). \tag{15}$$

This makes λ into an algebra homomorphism

$$\lambda : \mathbb{C}[G] \to \mathcal{B}(\ell_2(G)), \tag{16}$$

i.e. λ is a representation of the group algebra on $\ell_2(G)$. We define two new (closed) algebras via this representation. The reduced C^*-group algebra $C^*_{red}(G)$ of G is the closure of $\lambda(\mathbb{C}[G]) \subset \mathcal{B}(\ell_2(G))$ in the operator norm topology. The group von Neumman algebra of G, denote $L(G)$, is the closure of $\lambda(\mathbb{C}[G])$ in the strong operator topology on $\mathcal{B}(\ell_2(G))$.

One knows that $L(G)$ is a type II_1 von Neumann algebra, i.e. there is a trace on $L(G)$ defined by

$$\tau(a) := \langle e, ae\rangle \tag{17}$$

for $a \in L(G)$, where $e \in G$ is the identity element. An easy fact is that if G is an i.c.c. group, meaning that the conjugacy class of each $e \neq g \in G$ has infinite cardinality, then $L(G)$ is a factor, i.e. has trivial center. In particular this shows that if G is i.c.c. then $L(G)$ is a proper subalgebra of $B(H)$ (where we are writing H for the Hilbert space $\ell_2(G)$). Another fact is that if G is an amenable group, then $L(G)$ is the hyperfinite II_1 factor R.

Now consider the case where $G = \mathbb{F}_n$, the free group on n generators. Let us briefly recall the definition of $\mathbb{F}_n$ and some of its properties. Consider the set of all words, of arbitrary length, over the $2n+1$-letter alphabet $\{a_1, a_2, \dots, a_n, a_1^{-1}, a_2^{-1}, \dots \dots, a_n^{-1}\} \cup \{e\}$, where the letters of the alphabet satisfy no relations other than $ea_i = a_i e = a_i$, $ea_i^{-1} = a_i^{-1}e = a_i^{-1}$, $a_i^{-1}a_i = a_i a_i^{-1} = e$. We say that a word is reduced if its length cannot be reduced by applying one of the above relations. Then the set of all reduced words in this alphabet together with the binary operation of concatenating words and reducing constitutes the free group $\mathbb{F}_n$ on n generators. $\mathbb{F}_n$ is the group generated by n symbols satisfying no relations other than those required by the group axioms. $\mathbb{F}_n$ occurs in algebraic topology, where it is the fundamental group of a bouquet of n circles, i.e. n circles joined at a single point. Clearly $\mathbb{F}_1$ is isomorphic to the abelian group $\mathbb{Z}$, while $\mathbb{F}_n$ is non-abelian for $n > 1$ and in fact has trivial center. The integer n is called the rank of the free group; clearly $\mathbb{F}_n$ and $\mathbb{F}_m$ are isomorphic if and only if $m = n$.

Since F_n clearly has the infinite conjugacy class property, one knows that the group von Neumman algebra $L(\mathbb{F}_n)$ is a factor, called the free group factor. Murray and von Neumann showed that $L(\mathbb{F}_n)$ is not isomorphic to the hyperfinite factor, but otherwise nothing was known about the structure of those free group factors, when free probability was invented to understand them better.

While as pointed out above it is easy to see that $\mathbb{F}_n \simeq \mathbb{F}_m$ iff $m = n$, the corresponding problem for the free group factors is still unknown.

Free Group Factor Isomorphism Problem: Let $m, n \geq 2$, $n \neq m$. Are the von Neumann algebras $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ isomorphic?

The corresponding problem for the group C^*-algebras was solved by Pimsner and Voiculescu in 1982:

$$C^*_{red}(F_n) \not\simeq C^*_{red}(F_m) \text{ for } m \neq n. \tag{18}$$

There is the notion of *free product of groups.* If G, H are groups, then their free product $G * H$ is defined to be the group whose underlying set is the disjoint union of G and H, and which has the property that the only relations in $G * H$ are those inherited from G and H and the identification of the neutral elements of G and H. That is, there should be no non-trivial algebraic relations between elements of G and elements of H in $G \times H$. A more rigorous definition is the following: free product is the coproduct in the category of groups. Example: in the category of groups, the n-fold direct product of n copies of $\mathbb{Z}$ is the lattice $\mathbb{Z}^n$; the n-fold coproduct (free product) of n copies of $\mathbb{Z}$ is the free group $\mathbb{F}_n$ on n generators.

In the category of groups we can understand $\mathbb{F}_n$ via the decomposition $\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$. Is there a similar "free product of von Neumann algebras" that will help us to understand the structure of $L(\mathbb{F}_n)$? The notion of "freeness" aka "free independence" makes this precise. In order to understand what it means for elements in $L(G)$ to be "free" need to deal with infinite sums, so the algebraic notion of freeness will not do: we need a state.

1.3. Moments and Isomorphism of vN-algebras.

We will try to understand a vN-algebra with respect to a state. Let M be a vN-algebra and let $\phi : M \to \mathbb{C}$ be a state defined on M, i.e. a positive linear functional. Select finitely many elements $a_1, \dots, a_k \in M$.

Definition 1. *The collection of numbers gotten by applying the state to words in the alphabet* $\{a_1, \dots, a_k\}$ *is called the collection of joint moments of* $a_1, \dots, a_k$, *or the distribution of* $a_1, \dots, a_k$.

Definition 2. *The collection of numbers gotten by applying the state to words in the alphabet* $\{a_1, \dots, a_k, a_1^*, \dots, a_k^*\}$ *is called the collection of joint* $*$*-moments of* $a_1, \dots, a_k$, *or the* $*$*-distribution of* $a_1, \dots, a_k$.

Theorem 1. *Let* $M = vN(a_1, \dots, a_k)$ *be generated as von Neumann algebra by elements* $a_1, \dots, a_k$ *and let* $N = vN(b_1, \dots, b_k)$ *be generated as von Neumann algebra by elementes* $b_1, \dots, b_k$. *Let* $\phi : M \to \mathbb{C}$ *and* $\psi : N \to \mathbb{C}$ *be faithful normal states. If* $a_1, \dots, a_k$ *and* $b_1, \dots, b_k$ *have the same* $*$*-distributions with respect to* ϕ *and* ψ *respectively, then the map* $a_i \mapsto b_i$ *extends to a* $*$*-isomorphism of* M *and* N.

One proves the theorem by realizing that the assumptions imply that GNS-constructions with respect to ϕ and ψ are isomorphic.

Proposition 1.1. τ *on* $L(G)$ *is a faithful state.*

Proof. Suppose that $a \in L(G)$ satisfies

$$0 = \tau(a^*a) = \langle e, a^*ae\rangle = \langle ae, ae\rangle, \tag{19}$$

thus $ae = 0$. So we have to show $ae = 0 \implies a = 0$. To show that $a = 0$, it suffices to show that $\langle a\xi, \nu\rangle = 0$ for any $\xi, \nu \in \ell_2(G)$. It suffices to consider vectors of the form $\xi = g, \nu = h$ for $g, h \in G$, since we can get the general case from this by linearity and continuity. Now, by using the traciality of τ,

$$\begin{aligned}
\langle ag, h\rangle &= \langle age, he\rangle \\
&= \langle e, g^{-1}a^*he\rangle \\
&= \tau(g^{-1}a^*gh \\
&= \tau(a^*hg^{-1}) \\
&= \langle ae, hg^{-1}e\rangle \\
&= 0,
\end{aligned}$$

since the first argument to the inner product in the last line is 0. □

2. Freeness in the Free Group Factors

Definition 3. *Let* A *be a unital algebra and* $\phi : A \to \mathbb{C}$ *be a unital linear functional. Consider unital subalgebras* $A_1, \dots, A_n \subset A$. *We say that these subalgebras are free if*

$$\phi(a_1 \dots a_k) = 0 \tag{20}$$

whenever

- $\phi(a_i) = 0$, $i = 1, \dots, k$
- $a_1 \in A_{i(1)}, \dots, a_k \in A_{i(k)}$
- $i(j) \neq i(j+1)$ *for each* $j = 1, \dots, k-1$.

Note that this definition depends on the chosen state ϕ, so it is not an algebraic condition.

We will view the notion of freeness as an analogue of the classical notion of independence.

Let us say that a word in A is "alternating" with respect to the subalgebras $A_1, \ldots, A_n$ if adjacent elements come from different subalgebras. Then freeness says: the subalgebras $A_1, \ldots, A_n$ are free if any word in centred elements over these algebras which alternates is centred.

Proposition 2.1. *Let G be a group containing subgroups $G_1, \ldots, G_n$ such that $G = G_1 * \cdots * G_n$. Let τ be the state $\tau(a) = \langle e, ae\rangle$ on $\mathbb{C}[G]$. Then the algebras $\mathbb{C}[G_1], \ldots, \mathbb{C}[G_n]$ are free with respect to τ.*

Proof. Let $a_1 a_2 \ldots a_k$ be an element in $\mathbb{C}[G]$ which alternates with respect to the subalgebras $\mathbb{C}[G_1], \ldots, \mathbb{C}[G_n]$, and assume the factors of the product are centred with respect to τ. Since τ is the "coefficient of the identity" state, this means that if $a_j \in \mathbb{C}[G_{i(j)}]$, then a_j looks like

$$a_j = \sum_{g \in G_{i(j)}} a_j(g) g \tag{21}$$

and $a_j(e) = 0$. Thus we have

$$\tau(a_1 a_2 \ldots a_k) = \sum_{g_1 \in G_{i(1)} \cdots g_k \in G_{i(k)}} a_1(g_1) a_2(g_2) \ldots a_k(g_k) \tau[g_1 g_2 \ldots g_k]. \tag{22}$$

Now, $\tau[g_1 g_2 \ldots g_k] \neq 0$ only if $g_1 g_2 \ldots g_k = e$. $g_1 g_2 \ldots g_k$ is an alternating word in G with respect to the subgroups $G_1, G_2, \ldots, G_n$, and since $G = G_1 * G_2 * \cdots * G_n$, this can happen only when at least one of the factors, let's say g_j, is equal to e; but in this case $a_j(g_j) = a_j(e) = 0$. So each summand in the sum for $\tau[a_1 a_2 \ldots a_k]$ vanishes and we have $\tau[a_1 a_2 \ldots a_k] = 0$, as required. □

Freeness of the subgroup algebras $\mathbb{C}[G_1], \ldots, \mathbb{C}[G_n]$ is just a simple reformulation of the fact that $G_1, \ldots, G_n$ are free subgroups of G. However, a non-trivial fact is that this reformulation carries over to closures of the subalgebras.

Proposition 2.2. (1) *Let A be a C^*-algebra, $\phi : A \to \mathbb{C}$ a state. Let $B_1, \ldots, B_n \subset A$ be unital $*$-subalgebras which are free with respect to ϕ. Put $A_i := \overline{B_i}^{\|\cdot\|}$. Then $A_1, \ldots, A_n$ are also free.*

(2) *Let M be a vN-algebra, $\phi : M \to \mathbb{C}$ a normal state. Let $B_1, \ldots, B_n$ be unital $*$-subalgebras which are free. Put $M_i := vN(B_i)$. Then $M_1, \ldots, M_n$ are also free.*

The proof is left as an exercise.

3. Basic Properties of Freeness

We adopt the general philosophy of regarding freeness as a non-commutative analogue of the classical notion of independence in probability theory. Thus we refer to it often as "free independence."

In general we refer to a pair (A, ϕ) consisting of a unital algebra A and a unital linear functional $\phi : A \to \mathbb{C}$ as a non-commutative probability space.

If A is a C^*-algebra and ϕ a state, we have a C^*-probability space.

If A is a vN-algebra and ϕ is a faithful normal state, we have a W^*-probability space.

Proposition 3.1. *Let (B, ϕ) be a non-commutative probability space. Consider unital subalgebras $A_1, \dots, A_n \subset B$ which are free. Let A be the algebra generated by $A_1, \dots, A_n$. Then $\phi|_A$ is determined by $\phi|_{A_1}, \dots, \phi|_{A_n}$ and the freeness condition.*

Proof. Elements in the generated algebra A are linear combinations of words of the form $a_1 \dots a_k$ with $a_j \in A_{i(j)}$ for some $i(j) \in \{1, \dots, n\}$ which meet the condition that neighbouring elements come from different subalgebras. We neet to calculate $\phi(a_1 \dots a_k)$ for such words. Let us proceed in an inductive fashion.

We know how to calculate $\phi(a)$ for $a \in A_i$ for some $i \in \{1, \dots, n\}$.

Now suppose we have a word of the form $a_1 a_2$ with $a_1 \in A_{i(i)}$ and $a_2 \in A_{i(2)}$ with $i(1) \neq i(2)$. By the definition of freeness, this implies

$$\phi[(a_1 - \phi(a_1)1)(a_2 - \phi(a_2)1)] = 0. \tag{23}$$

But

$$(a_1 - \phi(a_1)1)(a_2 - \phi(a_2)1) = a_1 a_2 - \phi(a_2) a_1 - \phi(a_1) a_2 + \phi(a_1)\phi(a_2)1. \tag{24}$$

Hence we have

$$\phi(a_1 a_2) = \phi\big[\phi(a_2) a_1 + \phi(a_1) a_2 - \phi(a_1)\phi(a_2)1\big] = \phi(a_1)\phi(a_2). \tag{25}$$

Continuing in this fashion, we know that

$$\phi(a_1^c \dots a_k^c) = 0 \tag{26}$$

by the definition of freeness, where $a_i^c = a_i - \phi(a_i)1$ is a centred random variable. But then

$$\phi(a_1^c \dots a_k^c) = \phi(a_1 \dots a_k) + \text{lower order terms in } \phi, \tag{27}$$

where the lower order terms are already dealt with by induction hypothesis.

□

Definition 4. *Let (A, ϕ) be a non-commutative probability space. Elements $a_1, \dots, a_n \in A$ are said to be freely independent if the generated subalgebras $A_i = alg(1, a_i)$ are free in A with respect to ϕ.*

For example, if a, b are freely independent, then

$$\phi[(a - \phi(a)1)(b - \phi(b)1)] = 0 \implies \phi(ab) = \phi(a)\phi(b). \tag{28}$$

Slightly more complicated example: let $\{a_1, a_2\}$ be free from b. Then applying the state to the corresponding centred word:

$$\phi[(a_1 - \phi(a_1)1)(b - \phi(b)1)(a_2 - \phi(a_2)1)] = 0, \tag{29}$$

hence the linearity of ϕ gives

$$\phi(a_1 b a_2) = \phi(a_1 a_2)\phi(b). \tag{30}$$

A similar calculation shows that if $\{a_1, a_2\}$ is free from $\{b_1, b_2\}$, then

$$\phi(a_1 b_1 a_2 b_2) = \phi(a_1 a_2)\phi(b_1)\phi(b_2) + \phi(a_1)\phi(a_2)\phi(b_1 b_2) - \phi(a_1)\phi(a_2)\phi(b_1)\phi(b_2). \tag{31}$$

It is important to note that while free independence is analogous to classical independence, it is not a generalization of the classical case. Classical commuting random variables a, b are free only in trivial cases: $\phi(aabb) = \phi(abab)$, but the left hand side is $\phi(aa)\phi(bb)$ while the right hand side is $\phi(a^2)\phi(b)^2 + \phi(a)^2\phi(b^2) - \phi(a)^2\phi(b)^2$, which implies

$$\phi[(a - \phi(a))^2]\phi[(b - \phi(b))^2] = 0. \tag{32}$$

But then (note that states in classical probability spaces are always positive and faithful) one of the factors inside ϕ must be 0, so that one of a, b must be a scalar.

Observe that while freeness gives a concrete rule for calculating mixed moments, this rule is a priori quite complicated.

Proposition 3.2. *Let (A, ϕ) be a noncommutative probability space. The subalgebra of scalars $\mathbb{C}1$ is free from any other unital subalgebra $B \subset A$.*

Proof. Let $a_1 \dots a_k$ be an alternating word in centred elements of $\mathbb{C}1, B$. The case $k = 1$ is trivial, otherwise we have at least one $a_j \in \mathbb{C}1$. But then $\phi(a_j) = 0$ implies $a_j = 0$, so $a_1 \dots a_k = 0$. Thus obviously $\phi(a_1 \dots a_k) = 0$. $\square$