# Dynamics of Bose-Einstein Condensates

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# *INTERACTING MANY-BODY QUANTUM SYSTEMS*

$\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N}$ position of the particles.
Symmetric wave function: $\psi_N(x_1, \dots, x_N) \in L^2(\mathbb{R}^{3N})$

$$H_N = \sum_{j=1}^{N} \Big[ -\Delta_{x_j} + U(x_j) \Big] + \lambda \sum_{i<j} V(x_i - x_j)$$

$U$ is a one-body background ("trapping") potential
$V$ is the interaction potential

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \qquad i\partial_t \gamma_{N,t} = \Big[ H, \gamma_{N,t} \Big], \quad [A, B] = AB - BA$$

with $\gamma_{N,t} := |\psi_{N,t}\rangle\langle\psi_{N,t}|$ density matrix (1 dim. projection).

One particle density matrix:

$$\gamma_\psi^{(1)}(x, y) := \int \psi(x, x_2 \cdots x_N) \overline{\psi}(y, x_2' \cdots x_N') dx_2 \cdots dx_N dx_2' \cdots dx_N'$$

## *Time-independent BEC in Scaling Limit*

$$H_N = \sum_{j=1}^{N} \Big[ -\Delta_{x_j} + U(x_j) \Big] + \frac{1}{N} \sum_{i<j} N^3 V(N(x_i - x_j))$$

Approx Dirac delta interaction with range $1/N$ ("hard core")

[Dyson, Lieb-Seiringer-Yngvason, Lieb-Seiringer]

- Ground state energy is given by the Gross-Pitaevskii functional

$$\lim_{N\to\infty} \inf \operatorname{spec} \frac{H_N}{N} = \inf_{\varphi, \|\varphi\|=1} \mathcal{E}_{GP}(8\pi a_0, \varphi), \quad a_0 = \text{scatt. length of } V$$

$$\mathcal{E}_{GP}(\sigma, \varphi) := \int |\nabla \varphi|^2 + U|\varphi|^2 + \frac{\sigma}{2}|\varphi|^4$$

- Complete condensation in ground state:

$$\gamma_N^{(1)}(x; x') \to \phi(x)\overline{\phi(x')}, \qquad \phi = \text{minimizer of } \mathcal{E}_{GP}$$

## *Time Dependent GROSS-PITAEVSKII (GP) Theory*

The GP energy functional also describes the evolution:

$$\gamma^{(1)}_{N,0} \to \varphi(x)\bar{\varphi}(x') \quad \Longrightarrow \quad \gamma^{(1)}_{N,t} \to \varphi_t(x)\bar{\varphi}_t(x')$$

The condensate wave fn. evolves according to a NLS

$$i\partial_t \varphi_t = \Big[ -\Delta + U + 8\pi a_0 |\varphi_t|^2 \Big] \varphi_t, \qquad \varphi_{t=0} = \varphi$$

Many-body effects & corr $\to$ non-linear on-site self-interaction

Experiments of Bose-Einstein Condensation: Trap Bose gas and observe its evolution after the trap removed.

Dynamics: The ground state of trapped BEC is a highly excited state for the system without traps. GP describes also excited states and their evolution!

Cannot be completely correct. Now set $U = 0$.

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j} V_\beta(x_i - x_j), \quad V_\beta(x) := N^{3\beta} V(N^\beta x),\ 0 < \beta \le 1$$

*THEOREM:* [Erdős-Schlein-Y, 2008] Assume $V \ge 0$ and $V(x) \le C(1+|x|)^{-5}$. Suppose the initial state satisfies

$$\gamma_{N,0}^{(1)}(x,y) \to u_0(x)\bar{u}_0(y), \qquad u \in H^1(\mathbb{R}^3)$$

Then for every $k \ge 1$ and $t > 0$ fixed

$$\gamma_{N,t}^{(k)} \to |u_t\rangle\langle u_t|^{\otimes k} \qquad N \to \infty$$

$$i\partial_t u_t = -\Delta u_t + \sigma |u_t|^2 \phi_t, \qquad \sigma = \begin{cases} b_0 & \text{if} \quad 0 < \beta < 1 \\ 8\pi a_0 & \text{if} \quad \beta = 1 \end{cases}$$

where $a_0$ is the scatt. length of $V$ and $b_0 = \int \mathrm{d}x V(x) \neq 8\pi a_0$

Adami, Bardos, Golse, Teta: one dim result. Use $\delta \le -\Delta$ in $\mathbb{R}$ and the EY approach.

## SCATTERING LENGTH

$$\left(-\Delta + \frac{1}{2}V(x)\right)(1 - w(x)) = 0 \qquad \text{with } w(x) \to 0 \text{ for } |x| \to \infty\,.$$

$$w(x) = \frac{a_0}{|x|} \quad \text{for } |x| \to \infty \qquad \int \mathrm{d}x V(x)(1 - w(x)) = 8\pi a_0$$

Dyson's trial function for ground state:

$$W_N(\mathbf{x}) = \prod_{j<k} \left[1 - w(N(x_j - x_k))\right]$$

States with and without short range structure:

$$\psi_N(\mathbf{x}) = W_N(\mathbf{x}) \prod_{j=1}^{N} u_0(x_j), \qquad \phi_N = \prod_{j=1}^{N} u_0(x_j)$$

$$\lim_{N\to\infty} N^{-1}\langle \psi_N, H_N \psi_N\rangle = \int |\nabla u(x)|^2 + 4\pi a_0 |u(x)|^4$$

$$\lim_{N\to\infty} N^{-1}\langle \phi_N, H_N \phi_N\rangle = \int |\nabla u_0(x)|^2 + \frac{b_0}{2}|u(x)|^4$$

The theorem for $\beta = 1$ holds for $\psi_N$ and $\phi_N$.

Our Theorem shows that the local singular structure is preserved by the $N$-body evolution for initial state $\psi_N$. For product initial state, it shows that the local structure emerges .

$$i\partial_t \phi_{N,t} = H_N \phi_{N,t}, \ \phi_{N,t=0} = \phi_N$$

$$N^{-1}\langle \phi_{N,t}, H_N \phi_{N,t}\rangle = N^{-1}\langle \phi_N, H_N \phi_N \rangle$$

$$\to \mathcal{E}_{GP}(b_0, u_0) \neq \mathcal{E}_{GP}(8\pi a_0, u_0) = \mathcal{E}_{GP}(8\pi a_0, u_t)$$

For product initial state, the GP energy functional (with the coupling constant $8\pi a_0$) does not describe the energy of the $N$-body system . But the time dependent one particle density matrices in a weak limit is still given by the GP equation with coupling constant $8\pi a_0$.

Mathematically: The convergence of the time dependent density matrices is so weak that the energy does not converge.

Physically: For states with product initial data, the short scale behavior will show the characteristic $1 - w(N(x_i - x_j))$ structure after a short initial layer. This lowers the energy of the system locally. The energy lost was transfered to energy in other scales.

## *NONLINEAR HARTREE EQUATION:* $\beta = 0, V_\beta = V$

$$i\partial_t \varphi_t = -\Delta \varphi_t + \left(V \star |\varphi_t|^2\right)\varphi_t$$

Hepp: smooth potential, Ginibre-Velo: use coherent state. Schlein-Rodnianski apply to states with fix number of particles with Coulomb potential.

Spohn: bounded potential, BBGKY hierarchy.

Bardos, Golse and Mauser: convergence to hierarchy for Coulomb case, but no uniqueness nor a priori estimates.

Erdos-Y: Uniqueness of Coulomb case and a priori estimate.

bosonic star (Elgart-Schlein, Frohlich-Schwarz)

## Fundamental difficulty of $N$-particle analysis $(N \gg 1)$

There is no good norm. The conserved $L^2$-norm is too strong.

Let $\Psi = \otimes_1^N f$, $\Phi = \otimes_1^N g$, then $\|\Psi - \Phi\|^2 = 2 - 2\langle f, g\rangle^N \approx 2$.

Problem: $\psi(x_1, \ldots x_N)$ carries info of all particles (too detailed).

Keep only information about the $k$-particle correlations:

$$\gamma_\psi^{(k)}(X_k, X_k') := \int \psi(X_k, Y_{N-k})\overline{\psi}(X_k', Y_{N-k})dY_{N-k}$$

where $X_k = (x_1, \ldots x_k)$. It monitors only $k$ particles.

Quantum analog of the marginals of a probability density.

It is an operator acting on the $k$-particle space.

$$H = -\sum_{j=1}^{N} \Delta_j + \frac{1}{N} \sum_{j<k} V_\beta(x_j - x_k), \qquad i\partial_t \gamma_{N,t} = \left[H, \gamma_{N,t}\right]$$

**The BBGKY Hierarchy:** The family $\{\gamma_{N,t}^{(k)}\}_{k=1}^{N}$ satisfies

$$i\partial_t \gamma_{N,t}^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_{N,t}^{(k)}\right]$$

$$+ \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[V_\beta(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\right] + \text{lower order terms}\,.$$

$$\mathrm{Tr}_2 \left[V(x_1 - x_2), \gamma^{(2)}\right]$$

$$= \int \mathrm{d}x_2 \left(V(x_1 - x_2) - V(x_1' - x_2)\right) \gamma^{(2)}(x_1, x_2; x_1', x_2)\,.$$

## *Derivation of the Hartree equation:* $\beta = 0, V_\beta = V$

**Special case: $k = 1$:**

$$i\partial_t \gamma^{(1)}_{N,t}(x_1; x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\gamma^{(1)}_{N,t}(x_1; x_1')$$
$$+ \int \mathrm{d}x_2 \left(V(x_1 - x_2) - V(x_1' - x_2)\right) \gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2) + o(1)\,.$$

To get a closed equation for $\gamma^{(1)}_{N,t}$, we assume **Propagation of chaos:**

If initially $\gamma^{(2)}_{N,0} = \gamma^{(1)}_{N,0} \otimes \gamma^{(1)}_{N,0}$, then hopefully $\gamma^{(2)}_{N,t} \approx \gamma^{(1)}_{N,t} \otimes \gamma^{(1)}_{N,t}$.

Assume that $\gamma^{(1)}_{N,t} \to \omega_t$. With $\varrho_t(x) := \omega_t(x; x)$ . We then have the Hartree eq

$$i\partial_t \omega_t = \left[ -\Delta + V * \varrho_t\,,\ \omega_t \right]$$

**The Hartree Hierarchy $\beta = 0$:** As $N \to \infty$, the BBGKY hierarchy formally converges to the Hartree hierarchy

$$i\partial_t \gamma^{(k)}_{\infty,t} = \sum_{j=1}^{k} \left[ -\Delta_{x_j}, \gamma^{(k)}_{\infty,t} \right] + \sum_{j=1}^{k} \text{Tr}_{k+1} \left[ V(x_j - x_{k+1}), \gamma^{(k+1)}_{\infty,t} \right]$$

Remark:

$$\gamma^{(k)}_t (\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^{k} \phi_t(x_j) \overline{\phi_t}(x'_j) \qquad \left( \gamma^{(k)}_t = |\phi_t\rangle\langle\phi_t|^{\otimes k} \right)$$

is a solution of the Hartree hierarchy if $\phi_t$ satisfies

$$i\partial_t \phi_t = -\Delta \phi_t + (V * |\phi_t|^2) \phi_t \, .$$

## Strategy for Rigorous Derivation $\beta = 0$:

- Prove the compactness of $\{\gamma_{N,t}^{(k)}\}_{k=1}^{N}$ with respect to some weak topology

- Prove that the limit point $\{\gamma_{\infty,t}^{(k)}\}_{k\geq 1}$ is a solution of the infinite Hartree equation.

- Prove the apriori estimate needed for the uniqueness of the hierarchy.

- Prove the uniqueness (well-poseness) of the solution of the infinite Hartree hierarchy.

## Main Difficulties for Rigorous Derivation $\beta = 1$:

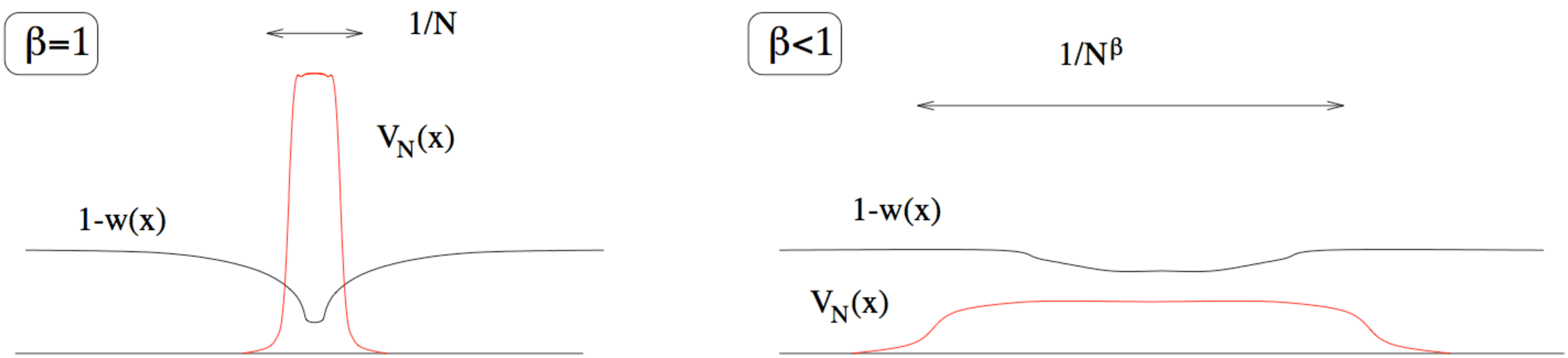
1. Derive the GP hierarchy: Suppose $\gamma_{N,t}^{(k)} \to \gamma_{\infty,t}^{(k)}$ as $N \to \infty$. Then:

$$i\partial_t \gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k} \left[ -\Delta_{x_j}, \gamma_{\infty,t}^{(k)} \right] + 8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[ \delta(x_j - x_{k+1}), \gamma_{\infty,t}^{(k+1)} \right]$$

## Emergence of the scattering length:

Suppose $\gamma^{(2)}_{N,t}(x_1,x_2;x'_1,x_2) = [1-w(N(x_1-x_2))]\gamma^{(1)}_{N,t}(x_1,x'_1)\gamma^{(1)}_{N,t}(x_2,x_2)$,

$$\int dx_2 V_\beta(x_1-x_2)[1-w(N(x_1-x_2))]f(x_2) = f(x_1)\times\begin{cases} 8\pi a_0 & \text{if } \beta=1 \\ b_0 & \text{if } 0<\beta<1 \end{cases}$$



2. Well-poseness of the GP hierarchy in the $H_1$ class:

$$\text{Tr}\,(1-\Delta_1)\dots(1-\Delta_k)\gamma^{(k)}_{\infty,t} \le C^k \qquad (\dagger)$$

Tool: Analysis on Feynman diagrams.

3. A priori estimate so that (†) holds. Due to the short scale structure, the estimate

$$\text{Tr}\,(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\gamma^{(k)}_{N,t} \le C^k$$

is wrong.

Only after taking the weak limit so that the short scale structure disappears, can such bounds hold.

Klainerman-Machedon: Uniqueness via space-time norm.

## Method of Moments of Energy—the second moment

**Proposition:** For any wave function $\psi_N$:

$$\langle \psi_N,\ H_N^2\, \psi_N \rangle \geq\ C N^2 \int \mathrm{d}\mathbf{x}\ |\nabla_1 \nabla_2\, \phi_{12}|^2$$

with $\phi_{12}(\mathbf{x}) = [1 - w(N(x_1 - x_2))]^{-1} \psi_N(\mathbf{x})$.

Consequence: any eigenfunction with energy $\simeq N$ must have the short scale structure $[1 - w(N(x_1 - x_2))]$ when $x_1$ is near $x_2$.

• BEC for ground state $\psi$ without scaling: Off-diagonal long range order (Yang)

• How to interpret systems with negative scattering length?

$$\inf_u \int |\nabla u|^2 \mathrm{d}x + \int U|u|^2 - 4\pi a_0 \int |u|^4 \mathrm{d}x = -\infty$$

The BEC cannot be the ground state! BEC for system with negative correlation length is a metastable state.

• fermi systems