Spectral gaps for periodic Schroedinger operators with magnetic wells

Yuri Kordyukov
Institute of Mathematics
Russian Academy of Sciences
Ufa, Russia

The setting

- M a noncompact oriented smooth manifold of dimension $n \geq 2$ such that $H^1(M,\mathbb{R}) = 0$ ($\Longleftrightarrow$ each closed one-form is exact).

- Γ a finitely generated, discrete group, which acts properly discontinuously on M so that M/Γ is a compact smooth manifold.

  EXAMPLE: $M = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$.

  EXAMPLE: M — the Poincaré upper-half plane, Γ — the fundamental group of a compact Riemann surface.

The setting

- $g = \sum_{i=1}^n \sum_{j=1}^n g_{ij}(x) dx^i dx^j$ a Γ-invariant Riemannian metric on M:

- $\mathbf{B} = \sum_{i<j} b_{ij}(x) dx^i \wedge dx^j$ a real-valued Γ-invariant closed 2-form on M.

- ASSUME: there exists a 1-form $\mathbf{A} = \sum_{i=1}^n a_i(x) dx^i$ on M such that

$$d\mathbf{A} = \mathbf{B} \Longleftrightarrow b_{ij} = \frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j}.$$

- g_{ij} and b_{ij} Γ-periodic, a_i, in general, NOT.

The magnetic Schrödinger operator

- The Schrödinger operator with magnetic potential $\mathbf{A}$ — a self-adjoint operator in $L^2(M)$:

$$H^h = (ih\,d + \mathbf{A})^*(ih\,d + \mathbf{A}), \quad h > 0.$$

- In $\mathbb{R}^n$, a self-adjoint operator in $L^2(\mathbb{R}^n, \sqrt{g}dx)$

$$H^h = \frac{1}{\sqrt{g}} \sum_{j,k} (ih\frac{\partial}{\partial x^j} + a_j(x)) \left[g^{jk}(x)\sqrt{g}(ih\frac{\partial}{\partial x^k} + a_k(x)) \right]$$

($g = \det(g_{ij})$, g^{jk} the inverse of g_{jk})

The magnetic Schrödinger operator

- In $\mathbb{R}^n$ with the standard Euclidean metric, a self-adjoint operator in $L^2(\mathbb{R}^n, dx)$

$$H^h = \sum_{j=1}^{n} (ih\frac{\partial}{\partial x^j} + a_j(x))^2.$$

THE MAGNETIC BOTTLES

The main problem

- A gap in the spectrum $\sigma(T)$ of a self-adjoint operator T is a maximal interval (a,b) such that

$$(a,b) \cap \sigma(T) = \emptyset$$

($\Longleftrightarrow$ a component of $\mathbb{R} \setminus \sigma(T)$)

PROBLEMS:

- Are there gaps in the spectrum of H^h in the semiclassical limit (as $h \to 0$)?
- Are there arbitrarily many number of gaps in the spectrum of H^h in the semiclassical limit (as $h \to 0$)?

Some more notation

- $B(x) : T_xM \to T_xM, x \in M$ the anti-symmetric linear operator:

$$g_x(B(x)u, v) = \mathbf{B}_x(u, v), \quad u, v \in T_xM.$$

- In local coordinates

$$B^i_j = \sum_{k=1}^{n} g^{ik} b_{kj} = \sum_{k=1}^{n} g^{ik} \left(\frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k} \right).$$

Even more notation

- The intensity of the magnetic field

$$\mathrm{Tr}^{+}(B(x)) = \frac{1}{2}\mathrm{Tr}\left([B^*(x)\cdot B(x)]^{1/2}\right).$$

- If $\pm i\lambda_j(x), j = 1, 2, \ldots, d, \lambda_j(x) > 0$, are the non-zero eigenvalues of $B(x)$, then

$$\mathrm{Tr}^{+}(B(x)) = \sum_{j=1}^{d} \lambda_j(x).$$

Magnetic wells

- DENOTE

$$b_0 = \min\{\mathrm{Tr}^{\,+}(B(x)) \,:\, x \in M\}.$$

- ASSUME:

there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\mathrm{Tr}^{\,+}(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}.$$

- EXAMPLE: $M = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n \Longrightarrow \mathcal{F} = (0,1)^n$ a fundamental domain.

Magnetic wells. II

- For any $\epsilon_1 \leq \epsilon_0$, let

$$U_{\epsilon_1} = \{x \in \mathcal{F} \,:\, \mathrm{Tr}^{\,+}(B(x)) < b_0 + \epsilon_1\}.$$

  - U_{ϵ_1} an open subset of $\mathcal{F}$ such that $U_{\epsilon_1} \cap \partial\mathcal{F} = \emptyset$;
  - For $\epsilon_1 < \epsilon_0$, $\overline{U_{\epsilon_1}}$ is compact and included in the interior of $\mathcal{F}$.

- Any connected component of U_{ϵ_1} with $\epsilon_1 < \epsilon_0$ — a magnetic well (attached to the effective potential $h \cdot \mathrm{Tr}^{\,+}(B(x))$).

Tunneling and localization in wells

- Fix arbitrary $\epsilon_1 < \epsilon_2 < \epsilon_0$.
- H^h_D the Dirichlet realization of H^h in $D = \overline{U_{\epsilon_2}}$ (has discrete spectrum).

THEOREM [B. Helffer, Yu. K., 2006] $\exists C, c, h_0 > 0\ \forall h \in (0, h_0]$

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \mathrm{dist}(\lambda, \sigma(H^h_D)) < Ce^{-c/\sqrt{h}}\},$$

$$\sigma(H^h_D) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \mathrm{dist}(\lambda, \sigma(H^h)) < Ce^{-c/\sqrt{h}}\}.$$

Quasimodes and spectral gaps

THEOREM: Let $N \geq 1$.

SUPPOSE $\mu_0^h < \mu_1^h < \ldots < \mu_N^h$ a subset of an interval $I(h) \subset [0, h(b_0 + \epsilon_1))$:

1. There exist constants $c > 0$ and $M \geq 1$ such that for any $h > 0$ small enough

$$\mu_j^h - \mu_{j-1}^h > ch^M, \quad j = 1, \ldots, N,$$
$$\operatorname{dist}(\mu_0^h, \partial I(h)) > ch^M, \quad \operatorname{dist}(\mu_N^h, \partial I(h)) > ch^M;$$

Quasimodes and spectral gaps

2. Each μ_j^h is an approximate eigenvalue of H_D^h:

$$\|H_D^h v_j^h - \mu_j^h v_j^h\| = \alpha_j(h)\|v_j^h\|,$$

where $v_j^h \in C_c^\infty(D)$ and $\alpha_j(h) = o(h^M)$ as $h \to 0$.

THEN

$\sigma(H^h) \cap I(h)$ has at least N gaps for any sufficiently small $h > 0$.

Quasimodes and spectral gaps: sketch of the proof

There exists $\lambda_j^h \in \sigma(H^h) \cap I(h), j = 0, 1, \ldots, N$

$$\lambda_j^h - \mu_j^h = o(h^M), \quad h \to 0.$$

For any $h > 0$ small enough, we have

$$\lambda_j^h - \lambda_{j-1}^h > ch^M, \quad j = 1, \ldots, N.$$

Quasimodes and spectral gaps: sketch of the proof

DENOTE

$N_h(\alpha,\beta)$ — the number of eigenvalues of H^h_D on an arbitrary interval $(h\alpha, h\beta)$.

LEMMA: For some C and h_0

$$N_h(\alpha,\beta)\leq Ch^{-n},\quad \forall h\in(0,h_0]\ .$$

Quasimodes and spectral gaps: sketch of the proof

- **LEMMA:** Let $M > 0$ and $c > 0$. There exist $C > 0$ and $h_1 > 0$ such that
  - IF α^h and β^h are two points in the spectrum of H^h on the interval $I(h)$ with $\beta^h - \alpha^h > ch^M$,
  - THEN for any $h \in (0, h_1]$, $\sigma(H^h) \cap (\alpha^h, \beta^h)$ has at least one gap of length $\geq Ch^{M+n}$.

- By this lemma, each interval $(\lambda_j^h, \lambda_{j+1}^h)$ contains at least one gap in the spectrum of H^h of length $\geq Ch^{M+n}$

- $\Longrightarrow$ The spectrum of H^h on the interval $I(h)$ has at least N gaps of length $\geq Ch^{M+n}$ for any h small enough.

The general case

THEOREM [B. Helffer, Yu. K., 2007]

- ASSUME: there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\mathrm{Tr}^{\,+}(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}.$$

- THEN: for any interval $[\alpha, \beta] \subset [b_0, b_0 + \epsilon_0]$ and for any natural N, there exists $h_0 > 0$ such that, for any $h \in (0, h_0]$,

$$\sigma(H^h) \cap [h\alpha, h\beta]$$

has at least N gaps.

The general case: sketch of the proof

- Fix some natural N. Choose some

$$b_0 < \mu_0 < \mu_1 < \ldots < \mu_N < b_0 + \epsilon_1.$$

- For any $j = 0, 1, \ldots, N$, take any $x_j \in D$ such that

$$\mathrm{Tr}^{+}(B(x_j)) = \mu_j.$$

The general case: construction of quasimodes

- Choose a local chart $f_j : U_j \to \mathbb{R}^n$ defined in a neighborhood U_j of x_j with local coordinates $X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^n$.

- Suppose that

  - $f_j(U_j)$ is a ball $B = B(0, r)$ in $\mathbb{R}^n$, $f_j(x_j) = 0$,
  - the Riemannian metric at x_j becomes the standard Euclidean metric on $\mathbb{R}^n$,
  - $\mathbf{B}(x_j) = \sum_{k=1}^{d_j} \mu_{jk} dX_{2k-1} \wedge dX_{2k}$.

The general case: construction of quasimodes

- Let φ_j be a smooth function on B such that

$$|\mathbf{A}(X) - d\varphi_j(X) - A_j^q(X)| \leq C|X|^2,$$

where $A_j^q(X) = \frac{1}{2}\sum_{k=1}^{d_j} \mu_{jk}\,(X_{2k-1}dX_{2k} - X_{2k}dX_{2k-1})$.

- Write $X'' = (X_{2d_j+1}, \ldots, X_n)$.

- Let $\chi_j \in C_c^\infty(D)$ supported in a neighborhood of x_j, and $\chi_j(x) \equiv 1$ near x_j.

The general case: construction of quasimodes

$v_j^h \in C_c^\infty(D)$ defined as

$$v_j^h(x) = \chi_j(x) \exp\left(-i\frac{\varphi_j(x)}{h}\right) \times$$
$$\times \exp\left(-\frac{1}{4h}\sum_{k=1}^{d_j} \mu_{jk}(X_{2k-1}^2 + X_{2k}^2)\right) \exp\left(-\frac{|X''|^2}{h^{2/3}}\right).$$

The general case: construction of quasimodes

- THEN
$$\|(H_D^h - h\mu_j)v_j^h\| \leq Ch^{4/3}\|v_j^h\|.$$

- So Theorem follows from the abstract result with
$$\mu_j^h = h\mu_j, \quad M = 1.$$

The general case: refined version

THEOREM [B. Helffer, Yu. K., 2008]

ASSUME:

- there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\mathrm{Tr}^{\,+}(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}.$$

- the rank of $\mathbf{B}$ is constant in an open set $U \subset M$

The general case: refined version

THEN: for any interval

$$[\alpha, \beta] \subset \mathrm{Tr}^+ B(U),$$

there exist $h_0 > 0$ and $C > 0$ such that

$$\sigma(H^h) \cap [h\alpha, h\beta]$$

has at least $[Ch^{-1/3}]$ gaps for any $h \in (0, h_0]$.

Discrete potential wells

THEOREM [B. Helffer, Yu. K. 2007] ASSUME

- $b_0 = 0$, and there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that
$$\mathrm{Tr}\,^+(B(x)) \geq \epsilon_0, \quad x \in \partial\mathcal{F};$$
- there exists a zero $\bar{x}_0$ of B, $B(\bar{x}_0) = 0$, such that $\exists C > 0$
$$C^{-1}|x - x_0|^k \leq \mathrm{Tr}\,^+(B(x)) \leq C|x - x_0|^k$$
for all x in some neighborhood of x_0 with some integer $k > 0$.

Discrete potential wells

THEN

for any natural N, there exist $C > 0$ and $h_0 > 0$ such that

$$\sigma(H^h) \cap [0, Ch^{\frac{2k+2}{k+2}}]$$

has at least N gaps for any $h \in (0, h_0)$.

Discrete potential wells: model operator

- ASSUME: $\bar{x}_0$ a zero of $\mathbf{B}$ such that, for all x in some neighborhood of x_0,

$$C^{-1}|x-x_0|^k \leq \mathrm{Tr}^{\,+}(B(x)) \leq C|x-x_0|^k.$$

- Write the 2-form $\mathbf{B}$ in the local coordinates

$$f : U(\bar{x}_0) \to f(U(\bar{x}_0)) = B \subset \mathbb{R}^n, \quad f(\bar{x}_0) = 0,$$

as

$$\mathbf{B}(X) = \sum_{1\leq l<m\leq n} b_{lm}(X)\, dX_l \wedge dX_m, \quad X = (X_1, \ldots, X_n) \in B.$$

Discrete potential wells: model operator

- $\mathbf{B}^0$ the 2-form in $\mathbb{R}^n$ with polynomial components

$$\mathbf{B}^0(X) = \sum_{1\leq l<m\leq n} \sum_{|\alpha|=k} \frac{X^\alpha}{\alpha!}\frac{\partial^\alpha b_{lm}}{\partial X^\alpha}(0)\, dX_l \wedge dX_m,$$

- $\exists \mathbf{A}^0$ a 1-form on $\mathbb{R}^n$ with polynomial components:

$$d\mathbf{A}^0(X) = \mathbf{B}^0(X), \quad X \in \mathbb{R}^n.$$

Discrete potential wells: model operator

- $K^h_{\bar{x}_0}$ a self-adjoint differential operator in $L^2(\mathbb{R}^n)$:

$$K^h_{\bar{x}_0} = (ih\,d + \mathbf{A}^0)^*(ih\,d + \mathbf{A}^0),$$

where the adjoints are taken with respect to the Hilbert structure in $L^2(\mathbb{R}^n)$ given by the flat Riemannian metric $(g_{lm}(0))$ in $\mathbb{R}^n$:

$$K^h_{\bar{x}_0} = \sum_{j,k} g^{jk}(0)\left(ih\frac{\partial}{\partial x^j} + a^0_j(x)\right)\left(ih\frac{\partial}{\partial x^k} + a^0_k(x)\right).$$

Discrete potential wells: construction of quasimodes

- For any $j \in \mathbb{N}$, let

$$K^h_{\bar{x}_0} w^h_j = h^{\frac{2k+2}{k+2}} \lambda_j w^h_j, \quad w^h_j \in L^2(\mathbb{R}^n).$$

- Let $\chi \in C^\infty_c(U(\bar{x}_0))$ equal 1 in a neighborhood of $\bar{x}_0$.

- Define

$$v^h_j(x) = \chi(x) w^h_j(x).$$

Discrete potential wells: construction of quasimodes

- We have

$$\| \left(H_D^h - h^{\frac{2k+2}{k+2}} \lambda_j \right) v_j^h \| \leq C_j h^{\frac{2k+3}{k+2}} \| v_j^h \|.$$

- For a given natural N, choose any $C > \lambda_{N+1}$.

- Then the result follows from the abstract theorem with

$$\mu_j^h = h^{\frac{2k+2}{k+2}} \lambda_j .$$

Discrete potential wells: spectral concentration

THEOREM [Yu. K. 2005]

ASSUME

- $b_0 = 0$, and there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that
$$\mathrm{Tr}^{+}(B(x)) \geq \epsilon_0, \quad x \in \partial\mathcal{F};$$
- For some integer $k > 0$, $B(x_0) = 0 \Rightarrow \exists C > 0$
$$C^{-1}|x - x_0|^k \leq \mathrm{Tr}^{+}(B(x)) \leq C|x - x_0|^k$$
for all x in some neighborhood of x_0.

Discrete potential wells: spectral concentration

THEN

there exists an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lambda_m \to \infty \text{ as } m \to \infty,$$

such that for any a and b with $\lambda_m < a < b < \lambda_{m+1}$,

$$[ah^{\frac{2k+2}{k+2}}, bh^{\frac{2k+2}{k+2}}] \cap \sigma(H^h) = \emptyset.$$

Hypersurface potential wells

ASSUME:

- $b_0 = 0$, and there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\mathrm{Tr}^{\,+}(B(x)) \geq \epsilon_0, \quad x \in \partial\mathcal{F};$$

- there is an open subset U of $\mathcal{F}$ such that the zero set of $\mathbf{B}$ in U is a smooth oriented hypersurface S, and, moreover, there are constants $k \in \mathbb{N}$ and $C > 0$ such that for all $x \in U$ we have:

$$C^{-1}d(x,S)^k \leq |B(x)| \leq Cd(x,S)^k\,.$$

Hypersurface potential wells: more notation

- N the external unit normal vector to S, and $\tilde{N}$ an arbitrary extension of N to a smooth vector field on U;

- $\omega_{0,1}$ the smooth one form on S defined, for any vector field V on S, by

$$\langle V, \omega_{0,1}\rangle(y) = \frac{1}{k!}\tilde{N}^k(\mathbf{B}(\tilde{N}, \tilde{V}))(y), \quad y \in S,$$

where $\tilde{V}$ is a C^∞ extension of V to U.

$\omega_{0,1}$ is the leading part of $\mathbf{B}$ at S

Hypersurface potential wells: more notation

- By assumption, we have

$$\omega_{0,1}(x) \neq 0, \quad x \in S.$$

- Denote

$$\omega_{\min}(B) = \inf_{x \in S} |\omega_{0,1}(x)| > 0.$$

Hypersurface potential wells: more notation

- For any $\alpha \in \mathbb{R}$, the self-adjoint second order differential operator in $L^2(\mathbb{R}, dt)$:

$$P(\alpha) = -\frac{d^2}{dt^2} + \left(\frac{1}{k+1}t^{k+1} - \alpha\right)^2.$$

- Denote by $\lambda_0(\alpha)$ the bottom of the spectrum of the operator $P(\alpha)$.
- One can show that

$$\hat{\nu} := \inf_{\alpha \in \mathbb{R}} \lambda_0(\alpha) > -\infty.$$

Hypersurface potential wells: the main result

THEOREM: [B. Helffer, Yu. K. 2008]

For any interval

$$(a,b) \subset (\hat{\nu}\,\omega_{\min}(B)^{\frac{2}{k+2}}, +\infty),$$

there exist $h_0 > 0$ and $C > 0$ such that

$$\sigma(H^h) \bigcap [h^{\frac{2k+2}{k+2}} a, h^{\frac{2k+2}{k+2}} b]$$

has at least $[Ch^{-\frac{2}{3(k+2)}}]$ gaps for any $h \in (0, h_0]$.

Hypersurface potential wells: model operator

- g_0 the Riemannian metric on S induced by g.
- One can assume that U is an open tubular neighborhood of S:

$$\Theta : (-\varepsilon_0, \varepsilon_0) \times S \xrightarrow{\cong} U,$$

such that $\Theta\big|_{\{0\}\times S} = \mathrm{id}$ and $(\Theta^* g - \tilde{g}_0)\big|_{\{0\}\times S} = 0$, where a Riemannian metric $\tilde{g}_0$ on $(-\varepsilon_0, \varepsilon_0) \times S$:

$$\tilde{g}_0 = dt^2 + g_0.$$

Hypersurface potential wells: construction of quasimodes

- By adding to $\mathbf{A}$ the exact one form $d\phi$, where ϕ is the function satisfying

$$N(x)\phi(x) = -\langle N, \mathbf{A}\rangle(x), \quad x \in U,$$
$$\phi(x) = 0, \quad x \in S,$$

we may assume that $\langle N, \mathbf{A}\rangle(x) = 0, x \in U$.

Hypersurface potential wells: construction of quasimodes

- $\omega_{0.0}$ the one form on S induced by $\mathbf{A}$:

$$\omega_{0.0} = i_S^* \mathbf{A}$$

where i_S is the embedding of S into M.

- $\omega_{0,1}$ the one form on S defined, for any vector field V on S, by

$$\langle V, \omega_{0,1} \rangle (y) = \frac{1}{k!} \tilde{N}^k (\mathbf{B}(\tilde{N}, \tilde{V}))(y), \quad y \in S,$$

where $\tilde{V}$ is a C^∞ extension of V to U.

Hypersurface potential wells: model operator

- DEFINE: $H^{h,0}$ is the self-adjoint operator in $L^2(\mathbb{R}\times S, dt\, dx_{g_0})$:

$$H^{h,0}=-h^2\frac{\partial^2}{\partial t^2}+\left(ihd+\omega_{0,0}+\frac{1}{k+1}t^{k+1}\omega_{0,1}\right)^*\left(ihd+\omega_{0,0}+\frac{1}{k+1}t^{k+1}\omega_{0,1}\right)$$

with Dirichlet boundary conditions.

- The operator $H^{h,0}$ has discrete spectrum.

Hypersurface potential wells: model operator

- H^h_D the unbounded self-adjoint operator in $L^2(D)$ given by the operator H^h in the domain $D=\overline{U}$ with Dirichlet boundary conditions.

- CLAIM: IF $\lambda^0(h)$ such that $\lambda^0(h)\leq Dh^{(2k+2)/(k+2)}$ is an approximate eigenvalue of $H^{h,0}$:

$$\|(H^{h,0}-\lambda^0(h))w^h\|\leq Ch^{(2k+3)/(k+2)}\|w^h\|,\quad w^h\in C^\infty_c(\mathbb{R}\times S),$$

  THEN $\lambda^0(h)$ is an approximate eigenvalue of H^h_D:

$$\|(H^h_D-\lambda^0(h))v^h\|\leq Ch^{(2k+3)/(k+2)}\|v^h\|,\quad v^h=(\Theta^{-1})^*w^h\in C^\infty_c(U).$$

Hypersurface potential wells: construction of quasimodes

- Take $x_1 \in S$ such that $|\omega_{0,1}(x_1)| = \omega_{\min}(B)$ $(= \inf_{x\in S} |\omega_{0,1}(x)|)$.
- Take normal coordinates $f : U(x_1) \subset S \to \mathbb{R}^{n-1}$ on S defined in a neighborhood $U(x_1)$ of x_1, where $f(U(x_1)) = B(0,r)$ is a ball in $\mathbb{R}^{n-1}$ centered at the origin and $f(x_1) = 0$.
- Choose a function $\phi \in C^\infty(B(0,r))$ such that $d\phi = \omega_{0,0}$.
- Write $\omega_{0,1} = \sum_{j=1}^{n-1} \omega_j(s)\, ds_j$.

Hypersurface potential wells: construction of quasimodes

- Consider $\alpha_1 \in \mathbb{R}$ such that $\lambda_0(\alpha_1) = \lambda\omega_{\min}(B)^{-2/(k+2)} \geq \hat{\nu}$.
- $\psi \in L^2(\mathbb{R})$ a normalized eigenfunction of $P(\alpha_1)$, corresponding to $\lambda_0(\alpha_1)$:

$$\left[-\frac{d^2}{dt^2} + \left(\frac{1}{k+1}t^{k+1} - \alpha_1\right)^2\right]\psi(t) = \lambda\omega_{\min}(B)^{-\frac{2}{k+2}}\psi(t), \quad \|\psi\|_{L^2(\mathbb{R})} = 1.$$

- Put

$$\Psi_h(t) = \omega_{\min}(B)^{\frac{1}{2(k+2)}} h^{-\frac{1}{2(k+2)}} \psi(\omega_{\min}(B)^{\frac{1}{k+2}} h^{-\frac{1}{k+2}} t).$$

Hypersurface potential wells: construction of quasimodes

$\Phi \in C^\infty(\mathbb{R} \times B(0,r))$ is defined by

$$\Phi_h(t,s) = ch^{-\beta/2(n-1)}\chi(s)\exp\left(-i\frac{\phi(s)}{h}\right)\exp\left(i\frac{\alpha_1}{\omega_{\min}(B)^{-\frac{k+1}{k+2}}h^{\frac{1}{k+2}}}\sum_{j=1}^{n-1}\omega_j(0)s_j\right)$$
$$\times\exp\left(-\frac{|s|^2}{2h^{2\beta}}\right)\Psi_h(t), \quad t\in\mathbb{R}, \quad s\in B(0,r),$$

where $\beta = \frac{1}{3(k+2)}$, $\chi \in C_c^\infty(B(0,r))$ is a cut-off function, and c is chosen in such a way that $\|\Phi\|_{L^2(S\times\mathbb{R})} = 1$.

Hypersurface potential wells: construction of quasimodes

- LEMMA:

  For any $\lambda \geq \hat{\nu}\, \omega_{\min}(B)^{2/(k+2)}$, we have

$$\|(H^{h,0} - \lambda h^{\frac{2k+2}{k+2}})\Phi_h\| \leq C h^{\frac{6k+8}{3(k+2)}} \|\Phi_h\|.$$

- Take

$$a < \lambda_0 < \lambda_1 < \ldots < \lambda_N < b.$$

- Then the result follows from the abstract theorem with

$$\mu_j^h = h^{\frac{2k+2}{k+2}} \lambda_j.$$

References

[1] Helffer B., Kordyukov Yu. A. Spectral gaps for periodic Schrödinger operators with hypersurface magnetic wells, preprint arXiv:0801.4460.

[2] Helffer B., Kordyukov Yu. A. The periodic magnetic Schrödinger operators: spectral gaps and tunneling effect, preprint math.SP/0702776; to appear in Proc. Steklov Inst. of Math., 261 (2008).

[3] Helffer B., Kordyukov Yu. A. Semiclassical asymptotics and gaps in the spectra of periodic Schrödinger operators with magnetic wells, Trans. Amer. Math. Soc. 360 (2008), 1681-1694.

[4] Kordyukov Yu. A. Spectral gaps for periodic Schrödinger operators with strong magnetic fields, Commun. Math. Phys. 253 (2005), no. 2, 371–384.