Workshop on computational and combinatorial commutative algebra
Fields Institute, Toronto, July 28, 2006

Tutorial 5a: Singularities of Schubert varieties

These exercises are intended to get you some idea of what these slices look like. Many common rational varieties end up making an appearance as one of these slices.

Two open problems are of course to find formulas for the multiplicities and Cohen-Macaulay types of points on Schubert varieties, either directly from the permutations or by characterizing the order ideals mentioned in the lecture. I have listed below some significant but possibly more manageable chunks of these problems.

Most of the content of the lecture is covered in the research-expository paper by myself and Alex Yong [AWAY]. We also have some code (available at Alex Yong's web page) you can look at or use for some of these exercises. The major exception is the Krattenthaler rule on multiplicity, for which [CK2] is the reference for the form of the rule I presented.

Exercise 1.1. Calculate the slices $\Omega_u^\circ \cap X_v$ for the following pairs of u and v:

The first two are the simplest singularities of Schubert varieties.

- $u = 2143$, $v = 4231$.
- $u = 1324$, $v = 3412$.

The next two are the next simplest.

- $u = 32154$, $v = 53241$.
- $u = 14325$, $v = 45312$.

These four are an interesting case of of how a singularity can become "worse" all by itself

- $u = 432165$, $v = 463152$.
- $u = 431265$, $v = 463152$.
- $u = 413265$, $v = 463152$.
- $u = 143265$, $v = 463152$.

The next three give another example, saying something about determinantal singularities.

- $u = 84327651$, $v = 87436521$.
- $u = 43218765$, $v = 87436521$.
- $u = 43218765$, $v = 84327651$.

Exercise 1.2. Find all x with an interval pattern embedding of $[1324, 3412]$ or $[2143, 4231]$ in $[x, 35867214]$.

Exercise 1.3. Use the Krattenthaler path counting theorem to show that interval pattern embeddings for Grassmannian permutations gives equality of multiplicities.

Exercise 1.4. Fix an integer $n \geq 4$. Let v be the permutation $n(n-2)(n-1)\cdots 2(n-1)1$. Let u be the permutation $(n-2)\cdots 1n(n-1)$. Convince yourself that $\Omega_u^\circ \cap X_v$ is the variety of rank at most 1 $2 \times n-2$ matrices. Calculate their multiplicity and Cohen-Macaulay type.

Exercise 1.5. Find all the places at which multiplicity on X_{456123} changes. (In other words, find all u such that the multiplicity at e_u is bigger than the multiplicity at $e_{u'}$ for all $u' \geq u$.

Exercise 1.6. Look at the Schubert variety X_{34512}. Calculate that it is singular at e_u if and only if $u \leq 13524$, $u \leq 31425$, or $u \leq 14325$. Now check that the neighborhoods of points e_u on X_{34512} come in three isomorphism classes: one for smooth points, one for $u = 13524, 31425, 14325$, and one for $u \leq 13425$. (Don't forget the factor of $X_u^\circ \cong \urcorner^{\ell(u)}$ since this question is about neighborhoods and not slices.) Unfortunately, the difference between the last two isomorphism classes cannot be detected using multiplicity (2 in both cases) or Cohen-Macaulay type (1 in both cases).

Hint. One way to distinguish these neighborhoods is by their singular loci.

Exercise 1.7. How many X_v for $v \in S_6$ have multiplicity 3 at the origin? Of these, how many do not have an interval embedding of $[32154, 53241]$, $[21543, 52431]$, $[13254, 35142]$, or $[21435, 42513]$? (These are the four cases of multiplicity 3 for $n = 5$.

***Exercise 1.8.** Extend the Krattenthaler path counting theorem to covexillary (3412 avoiding) permutations. The Gröbner basis theorem of Knutson, Miller, and Yong [AKEMAY] might be helpful. (Their notation is upside down from ours.) Your sets of paths will likely be different from but equivalent to his for Grassmannian permutations.

***Exercise 1.9.** Find u, v, x, w such that $[u, v]$ and $[x, w]$ are isomorphic as intervals, but the multiplicity of X_v at e_u is different from the multiplicity of X_w at e_x. Do the same for Cohen-Macaulay type. Clearly, the isomorphism cannot come from an interval pattern embedding. The statement that there is no such example for Kazhdan-Luzstig polynomials is a well-known conjecture, now almost solved (combinatorially, without little reference to Schubert varieties) by Brenti and various coauthors.

***Exercise 1.10.** Characterize the singularities of Cohen-Macaulay type 3 or higher.

Related References

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