COMBINATORIAL AND COMPUTATIONAL COMMUTATIVE ALGEBRA: MONDAY AFTERNOON TUTORIAL

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1. Which of the rank conditions $r_\pi(i,j)$ essential (i.e., a minimal sublist of the $n \times n$ conditions that implies the others).

2. Write a `Macaulay 2` procedure that takes as input a permutation in S_n and checks whether the defining generators of I_π form a Gröbner basis, for diagonal and antidiagonal term orders. State a condition on π for when it's true. (You can use Q1, but it isn't essential.)

3. Using Q2 (or otherwise), determine the facets, multidegree and Hilbert series of the ideal of 2×2 minors of a generic 3×3 matrix, and of $\Delta_{I_{1357246}}$, in terms of semistandard Young tableau. Prove these simplical complex are homeomorphic to balls via a vertex decomposition. Give a manifestly positive combinatorial rule for the h-vector. Generalize as far as you can.

4. Using Q2 (or otherwise), guess/prove a rule for the facets of $\Delta_{\mathrm{in}_\prec(I_\pi)}$ where $\prec$ is antidiagonal term order.

5. Let B_-, B_+ be respectively the lower and upper triangular matrices acting on M_n =the space of $n \times n$ matrices by left and right multiplication. Prove that any $B_- \times B_+$ invariant subvariety of M_n is a set-theoretic union of matrix Schubert varieties.

6. Fix $\pi \in S_n$ and consider the south, then eastmost variable that appears in any of the defining minors. Do a one step degeneration on that variable. Describe (set-theoretically) the cone and projection pieces in terms of π (hint: use Q5). Is this always a geometric vertex decomposition?

7. Use your answer to Q7 to describe a manifestly positive way to compute the Schubert polynomial $\mathfrak{S}_\pi$. (This gives geometric meaning to the *transition formula* for these polynomials.)

8. (Open) Prove or disprove that in Q6, the defining generators are Gröbner enough.

9. Give an algorithm for checking if the generators of an ideal are Gröbner enough.

10. (Open) Give an *efficient* (polynomial-time) algorithm for checking if the generators of an ideal are Gröbner enough.

11. (Open-ended) Find other ways to degenerate matrix Schubert varieties. Here's one reason to care: such degenerations should lead to other formulas for Schubert polynomials (which is interesting to me in and of itself). But I mainly am interested in possible applications to the **Schubert problem**:

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Schubert polynomials $\mathfrak{S}_{w\pi}$ for $\pi \in S = \bigcup_{n\geq 1} S_n$ form a basis for polynomials in $\mathbb{Z}[x_1, x_2, \ldots]$. The structure constants $c^{\rho}_{\pi,\sigma} \in \mathbb{N}$ defined by

$$\mathfrak{S}_\pi \cdot \mathfrak{S}_\sigma = \sum_{\rho \in S_\infty} c^{\rho}_{\pi,\sigma} \mathfrak{S}_\rho,$$

are the **Schubert intersection numbers** counting triple intersections of generic translates of the Schubert varieties $\mathfrak{X}_\pi$, $\mathfrak{X}_\sigma$ and $\mathfrak{X}_{w_0\rho}$. It is a well-known objective to find an informative positive combinatorial rule for these numbers.

For the case π, σ and ρ are **Grassmannian** (the permutations has only one descent), such as in Q3, these numbers are the **Littlewood-Richardson coefficients** which govern both tensor product multiplicities of $GL_n(\mathbb{C})$ irreps and Schubert calculus of Grassmannians. There, a rule is known, and a good understanding of the formula for the Schur polynomials (namely, the tableau formula presented in the morning lecture) leads to it, and its proof.

It is reasonable to suspect that a "right" combinatorial rule for expressing the Schubert polynomials will be similarly successful. My hope is that this rule will be seen through Gröbner degeneration techniques.

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