“Gröbner degeneration, monomial ideals and simplicial complexes”

Lecture 1 for the

Workshop on Computational and Combinatorial Commutative Algebra

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Objectives of this lecture

To introduce:

- the principle of **Gröbner degeneration**;
- combinatorics of (squarefree) monomial ideals and simplicial complexes;
- basics of `Macaulay 2` usage;

(Generalized) determinantal ideals and symmetric polynomials will provide the main examples to illustrate the above ideas.

Gröbner bases

- Let $\mathbb{k}$ be a field, and $\mathbb{k}[\mathbf{x}]$ be a polynomial ring in $\mathbf{x} = (x_1, \ldots, x_N)$.
- Fix an ideal $I \subseteq \mathbb{k}[\mathbf{x}]$. Study: properties of I, or geometrically, those of $X = \mathrm{Spec}(\mathbb{k}[\mathbf{x}]/I)$.
- Pick a **term order** (total order) $\prec$ on monomials in $\mathbb{k}[\mathbf{x}]$.
- Let $\mathrm{LT}_\prec(f)$ denote the largest $\prec$ term in f.
- Call $\mathrm{in}_\prec(I) := \langle \mathrm{LT}_\prec(f) : f \in I \rangle$ the **initial (monomial) ideal.**
- A generating set $\{g_1, \ldots, g_k\}$ for I is a **Gröbner basis** if $\mathrm{in}_\prec(I) = \langle \mathrm{LT}_\prec(g_1), \ldots, \mathrm{LT}_\prec(g_k) \rangle$

Gröbner degeneration theory I

Picking $\prec$ chooses a **flat degeneration** of X to the initial scheme $\mathrm{in}_{\prec}(X) := \mathrm{Spec}(\mathbb{k}[\mathbf{x}]/\mathrm{in}(I))$, a schemey union of coordinate subspaces in $\mathbb{k}^N$.

Some properties of X are invariant under Gröbner degeneration.

- Main example of this lecture: Hilbert series (more later)

In general, properties "get worse" (**upper semicontinuity**)

- e.g., reducedness, Cohen-Macaulayness, Gorensteinness

<u>Moral:</u> Degenerating to monomial ideals can be useful since they are well understood, though sometimes it's too drastic.

Gröbner degeneration theory II: Gröbner enough bases

Along the way to full Gröbner degenerations, consider **partial step degenerations.**

- Pick a **partial term order** $\prec$ on monomials.

<u>Example:</u> Weigh monomials by the exponent of a variable $y = x_j$

- The **leading form** $\mathrm{LF}_{\prec}(f)$ is the sum of highest $\prec$ terms
- Extend the definition of the **initial (form) ideal**.
- A generating set is **Gröbner enough** if its leading forms generate the initial form ideal.
- Gröbner bases are necessarily Gröbner enough, but not conversely.

Computing and verifying Gröbnerness

We can prove a generating set is a Gröbner basis by, e.g.,

- directly from the definition
- `Macaulay 2`/Buchberger's algorithm

Open Problem: Give an efficient (e.g., polynomial-time algorithm) to check if a generating set is Gröbner enough for a partial term order $\prec$.

Presumably the solution will generalize Buchberger's famous algorithm. The case of the "exponent of y" partial term order is already interesting.

Main ideal example: Determinantal ideals I

These ideals (and their generalizations) are of classical and current interest (e.g., combinatorial representation theory and algebraic geometry).

Fix a generic matrix of $N := m \times n$ indeterminates:

$$Z := \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}.$$

$I_k := \langle (k+1) \times (k+1)$ minors of $Z \rangle$ is a **determinantal ideal** of $\Bbbk[Z]$.

<u>Example:</u> $I_1 = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{23} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix}, \ldots \right\rangle.$

$X_k := \mathrm{Spec}(\Bbbk[Z]/I_k)$ is the space of rank$\leq k$ matrices.

Main ideal example: Determinantal ideals II

Questions we might have about I_k:

- Is I_k radical?
- What is a **combinatorial formula** for its Hilbert series?
- Gröbner bases of I_k?

Fact: The defining minors of I_k are a Gröbner basis with respect to any **diagonal term order**, i.e., one that picks the diagonal term of a minor as its leading term.
Since $\mathrm{in}_{\prec}(I_k)$ is radical, so is I_k.

In our example: $\mathrm{in}_{\prec}(I_1) = \langle z_{11}z_{22}, z_{11}z_{23}, z_{12}z_{23}, \ldots \rangle$

(Vague) Question: What is the combinatorics of the generators of $\mathrm{in}_{\prec}(I_k)$?

Macaulay 2, a first look I

```
Macaulay 2, version 0.9.2
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : R=QQ[x11,x12,x13,x21,x22,x23,x31,x32,x33,
                               MonomialOrder=>Lex]
o1 = R
o1 : PolynomialRing
i2 : G = matrix{{x11,x12,x13},{x21,x22,x23},
                                 {x31,x32,x33}};
i3 : I = minors(2,G) -- built in procedure to
                        list (2x2) minors
```

Macaulay 2, a first look II

```
o3 = ideal (- x12*x21 + x11*x22, - x12*x31 + x11*x32,
            - x22*x31 + x21*x32, - x13*x21 + x11*x23,
            - x13*x31 + x11*x33, - x23*x31 + x21*x33,
            [...snip...]
i4 : I_0  -- look at 1st element of I
o4 = x11*x22 - x12*x21
o4 : R
i5 : leadTerm(I_0)
o5 = x11*x22
o5 : R
i6 : leadTerm I
o6 = | x22x33 x21x33 x21x32 x12x33 x12x23 x11x33 x11x32 x11x23 x11x22 |
              1       9
o6 : Matrix R  <--- R
```

Macaulay 2, a first look III

```
i7 : J=gb I
o7 = | x22x33-x23x32 x21x33-x23x31 x21x32-x22x31
        x12x33-x13x32 x12x23-x13x22 [...snip...] |
o7 : GroebnerBasis
i8 : A=ideal leadTerm gens J
o8 = ideal (x22*x33, x21*x33, x21*x32, x12*x33, x12*x23, x11*x33, x11*x32, x11*x23, x11*x22)
o8 : Ideal of R
i9: B=ideal leadTerm I
o9 = ideal (x22*x33, x21*x33, x21*x32, x12*x33, x12*x23, x11*x33, x11*x32, x11*x23, x11*x22)
i10 : A==B  -- were I's generators already Grobner?
o10 = true
```

(Squarefree) monomial ideals and simplicial complexes

I is a **monomial ideal** if

- it is generated by monomials $\mathbf{x}^\alpha$, $\alpha \in \mathbb{N}^N$; or
- $f = \sum_{\alpha \in \mathbb{N}^N} c_\alpha \mathbf{x}^\alpha \in I \implies \mathbf{x}^\alpha \in I$ whenever $c_\alpha \neq 0$;

A monomial ideal is **squarefree** if each $\mathbf{x}^\alpha \in I$ has $\alpha \in \{0,1\}^N$.

Squarefree monomial ideals have associated combinatorics in terms of (abstract) **simplicial complexes**, i.e., a collection Δ of subsets (faces) of a vertex set $\{1, 2, \ldots, N\}$, closed under taking subsets.

- if $\sigma \in \Delta$ and $\mid \sigma \mid = k+1$ then σ has **dimension** k
- $\dim(\Delta) =$maximum of the dimensions of faces of Δ

The Stanley-Reisner correspondence I

The Stanley-Reisner correspondence is a bijection:

$$\{\text{square-free monomial ideals}\} \leftrightarrow \{\text{simplicial complexes}\}$$

More specifically the **Stanley-Reisner ideal** of Δ is

$$I_\Delta := \langle \mathbf{x}^\sigma : \sigma \notin \Delta \rangle,$$

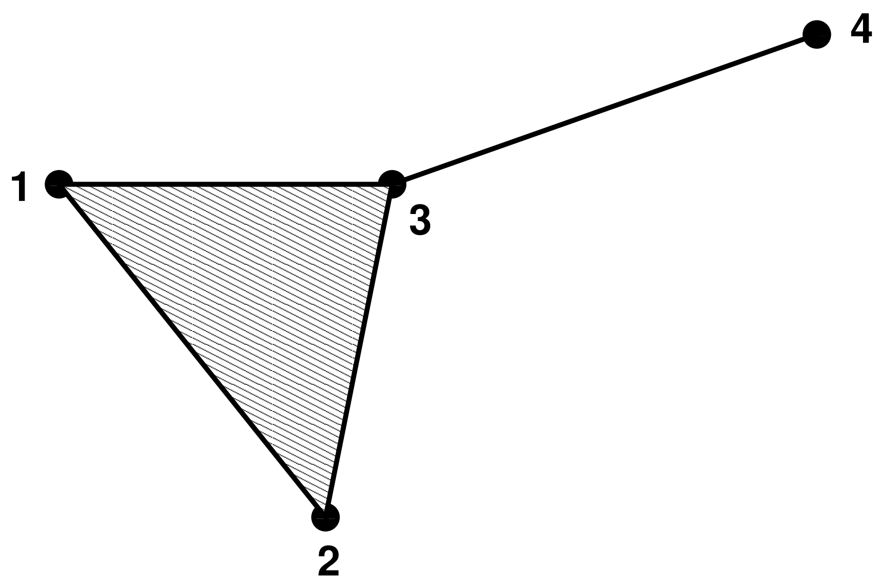
generated by monomials associated to **nonfaces** of Δ.

The **Stanley-Reisner ring** is the quotient $\mathbb{k}[\mathbf{x}]/I_\Delta$.
Consider the monomial prime ideal $\mathfrak{m}^\sigma$ generated by $x_i : i \in \sigma$. Let $\sigma^c = \{1, 2, \ldots, n\} \setminus \sigma$.

Fact: $I_\Delta = \bigcap_{\sigma \in \Delta} \mathfrak{m}^{\sigma^c}$

The Stanley-Reisner correspondence II

Example: Consider the following simplicial complex Δ



- $I_\Delta = \langle x_1x_4,\ x_2x_4\rangle = \langle x_1, x_2\rangle \cap \langle x_4\rangle$
- "Minimal poisonings": $\{x_4 = 0\} \leftrightarrow \{1, 2, 3\}$, $\{x_1, x_2 = 0\} \leftrightarrow \{3, 4\}$
- "Non-minimal poisonings", e.g., $\{x_1, x_2, x_4 = 0\} \leftrightarrow \{3\}$

The Stanley-Reisner correspondence III

Example (cont): Easy conclusions:

- Poisonings of I_Δ are in bijection with faces of Δ
- **minimal** poisonings are in bijection with **facets** (maximal faces) of Δ
- Poisoning A is a subface of Poisoning B iff $B \subseteq A$

The Stanley-Reisner correspondence IV

<u>Example:</u> Analyze $\Delta = \Delta_{\mathrm{in}_{\prec}(I_1)}$ for the initial ideal of the determinantal ideal of 2×2 minors of $Z = Z_{3\times 3}$.
Which variables $z_{ij} = 0$ can be encoded as fillings of a 3×3 matrix with "$+$"s: $\begin{array}{|c|c|c|}\hline + & + & \\ \hline + & & \\ \hline & & + \\ \hline\end{array}$

Minimal poisonings? (Δ is pure dimensional):

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & + & + \\ \hline & + & + \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline + & & \\ \hline & & + \\ \hline & + & + \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline + & + & \\ \hline & & \\ \hline & + & + \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline + & & \\ \hline + & & + \\ \hline & & + \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline + & + & \\ \hline + & & \\ \hline & & + \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline + & + & \\ \hline + & + & \\ \hline & & \\ \hline\end{array},$$

are in bijection with **semistandard Young tableaux**:

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}.$$

The Stanley-Reisner correspondence V

Thus facets of Δ are in bijection with semistandard Young tableau. These are basic objects of the:

- theory of symmetric functions
- representation theory of S_n
- representation theory of $GL_n(\mathbb{C})$
- Schubert calculus of Grassmannians

<u>Questions:</u>

- What about lower dimensional faces?
- Topology of Δ? (It's a ball...)
- Generality?
- So what? (...useful to derive Hilbert series formulas.)

Hilbert series I

If M is a $\Bbbk[\mathbf{x}]$-module, it is $\mathbb{N}^N$**-graded** if

- $M = \bigoplus_{\alpha \in \mathbb{N}^N} M_\alpha$
- $\mathbf{x}^\beta M_\alpha = M_{\beta+\alpha}$.

Assume $\dim_\Bbbk(M_\alpha) < \infty$ for all α. Then the $\mathbb{N}^N$-**graded Hilbert series** is

$$H(M, \mathbf{x}) = \sum_{\alpha \in \mathbb{N}^N} \dim_\Bbbk(M_\alpha)\, \mathbf{x}^\alpha.$$

Main example: $H(\Bbbk[\mathbf{x}]/I, \mathbf{x}) =$ sum of monomials not in I, for a homogeneous ideal I

Fact: Gröbner degenerations preserve Hilbert series, so it suffices to study the $I =$monomial ideal case.

Hilbert Series II

In particular for a Stanley-Reisner ring:

$$
\begin{aligned}
H(S/I_\Delta, \mathbf{x}) &= \sum \{\mathbf{x}^\alpha : \alpha \in \mathbb{N}^N \text{ and support}(\alpha) \in \Delta\} \\
&= \sum_{\sigma \in \Delta} \sum \{\mathbf{x}^\alpha : \alpha \in \mathbb{N}^N \text{ and support}(\alpha) = \sigma\} \\
&= \sum_{\sigma \in \Delta} \prod_{i \in \sigma} \frac{x_i}{1 - x_i}.
\end{aligned}
$$

<u>Exercise:</u> Compute the $\mathbb{N}^N$-graded Hilbert series for $\mathbb{k}[Z]/\mathrm{in}_\prec(I_1)$.

Macaulay 2 and hilbertSeries

Setting $x_i = t$ for all i, we obtain more coarsely graded Hilbert series:

```
i11 : hilbertSeries(R/I)

                2        3       4      6
      1 - 9$T  + 16$T  - 9$T  + $T
o11 = -----------------------------
                          9
                 (1 - $T)

o11 : Divide
```

K-polynomials

Fact: Every Hilbert series has the form

$$H(M;\mathbf{x}) = K(M;\mathbf{x})/(1-x_1)\cdots(1-x_N),$$

where $K(M;\mathbf{x})$ is the K-**polynomial.**

Theorem: For the Stanley-Reisner ring $\mathbb{k}[\mathbf{x}]/I_\Delta$:

$$K(\mathbb{k}[\mathbf{x}]/I_\Delta,\mathbf{x}) = \sum_{\sigma\in\Delta}\left(\prod_{i\in\sigma} x_i \prod_{j\notin\sigma}(1-x_j)\right).$$

Let $f_i = \#i$-dimensional faces of Δ; $(f_{-1}, f_1, \ldots, f_{\dim(\Delta)})$ is called the $f-\mathbf{vector}$ of Δ.

f, h-vectors of Δ

Corollary: Let $D := \dim(\Delta) + 1$.

$$H(\mathbb{k}[\mathbf{x}]/I_\Delta; t, t, \ldots, t) = \frac{1}{(1-t)^n} \sum_{i=0}^{D} f_{i-1} t^i (1-t)^{n-i}.$$

Cancelling $(1-t)^{n-D}$ from the sum and denominator:

$$\frac{1}{(1-t)^D} \sum_{i=0}^{D} f_{i-1} t^i (1 - t)^{D-i} = \frac{h_0 + h_1 t + \ldots + h_D t^D}{(1-t)^D}.$$

$(h_0, \ldots, h_D)$ is the $h-$**vector**.

Gröbner geometry of vertex decompositions I

Simplicial complex operations can have useful Gröbner analogues.

Fix a vertex $v \in \Delta$:

- the **star** of v is the closure of the union of faces containing v

$$\mathrm{star}(v) := \{\sigma \in \Delta : \sigma \cup \{v\} \in \Delta\}$$

- The star is the cone from v over the **link** of v

$$\mathrm{link}(v) := \{\sigma \in \Delta : v \notin \sigma, \sigma \cup \{v\} \in \Delta\}$$

- The **deletion** of v

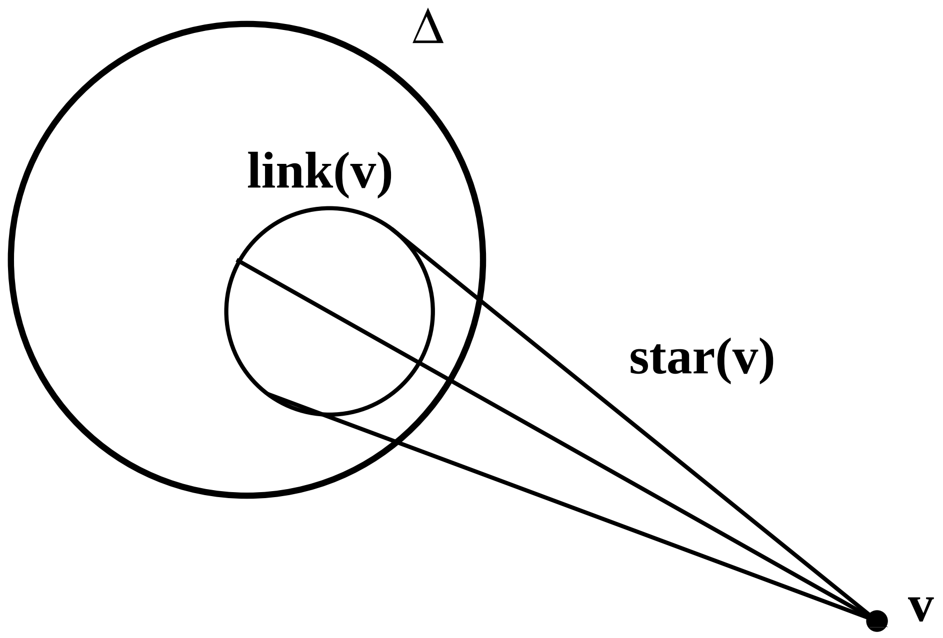
$$\mathrm{del}(v) := \{\sigma \in \Delta : v \notin \sigma\}$$

Gröbner geometry of vertex decompositions II

vertex decomposition: $\Delta = \mathrm{del}(v) \cup_{\mathrm{link}(v)} \mathrm{cone}(\mathrm{link}(v))$

If $\mathrm{del}(v) \cong d$-ball, and $\mathrm{link}(v) \cong (d-1)-$ball on the spherical surface of $\mathrm{del}(v)$, then Δ is homeomorphic to a d-ball.

Hence vertex decompositions are useful to prove ballness inductively.



Gröbner geometry of vertex decompositions III

Other useful facts to know:

Vertex decompositions give rise to **shellings** of a pure Δ, i.e., an ordering $\sigma_1, \sigma_2, \ldots, \sigma_k$ of the facets of Δ such that $\sigma_i \cap (\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_{i-1})$ is pure of dimension $d-1$. This implies there is a **new face** $\nu_i \in \sigma_i$ not in $\sigma_1, \ldots, \sigma_{i-1}$.

Δ shellable $\implies h_j \in \mathbb{N}$ counts ν_i's of $\dim(\nu_i) = j$

Gröbner geometry of vertex decompositions IV

Further examination of the geometry of the "exponent y" partial degeneration:

Let X be a closed subscheme of a vector space $V = H \oplus L$ (for **hyperplane** and **line**).

Let $\Bbbk^\star$ act on X by rescaling the first coordinate:

$$t \cdot (y, \vec{x}) := (ty, \vec{x}).$$

The flat limit

$$X' := \lim_{t \to 0} t \cdot X$$

is a "slow-motion" projection of X onto H.

Clearly, X' contains the actual projection Π of X onto H.

But it actually contains more.

Gröbner geometry of vertex decompositions V

Consider the closure $\overline{X}$ inside $H \times (L \cup \{\infty\})$.
Define $\Lambda := \overline{X} \cap (H \times \{\infty\}) \subseteq H \times \{\infty\} \cong H$.

<u>Theorem/Definition:</u> The support of the limit scheme X' is

$$(\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times L),$$

a union of the **projection piece** and the **cone piece.**
X is homogeneous implies a bound

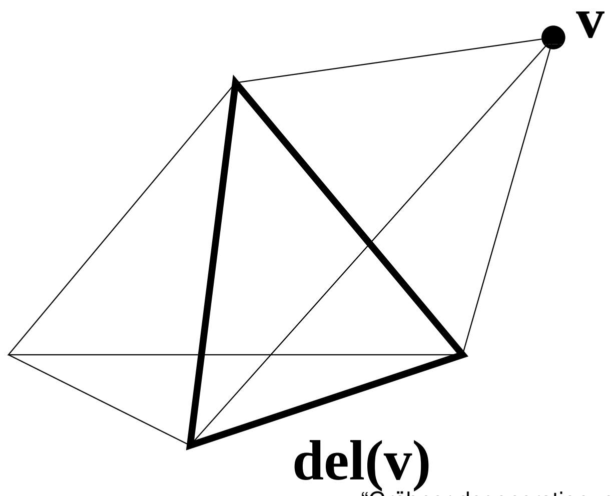
$$H(X,t) \geq H(\Pi,t) + \frac{t}{1-t} H(\Lambda,t),$$

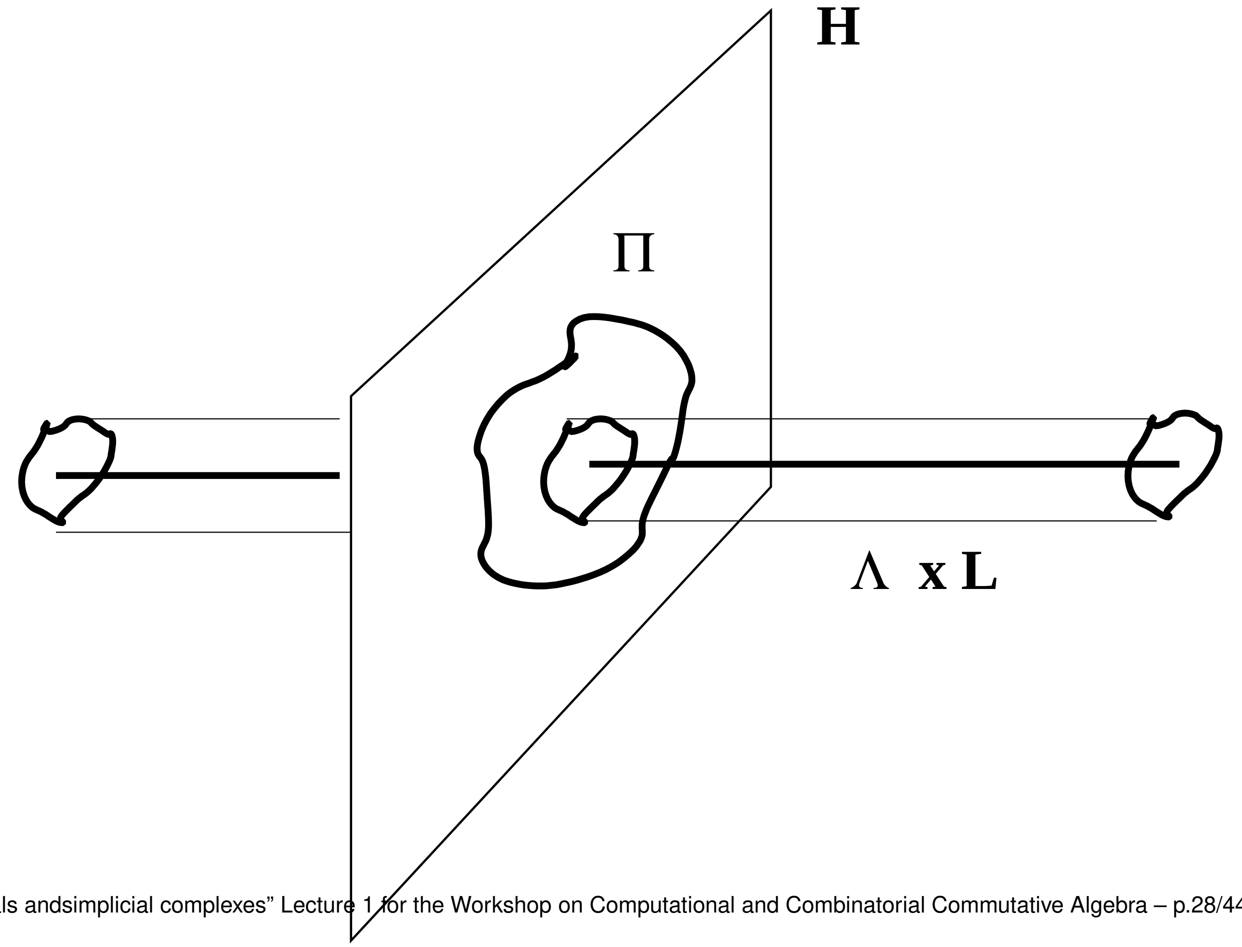
with "$=$" iff the set-theoretic equality for X' is also scheme-theoretic (a **geometric vertex decomposition**).

Gröbner geometry of vertex decompositions VI

This relates

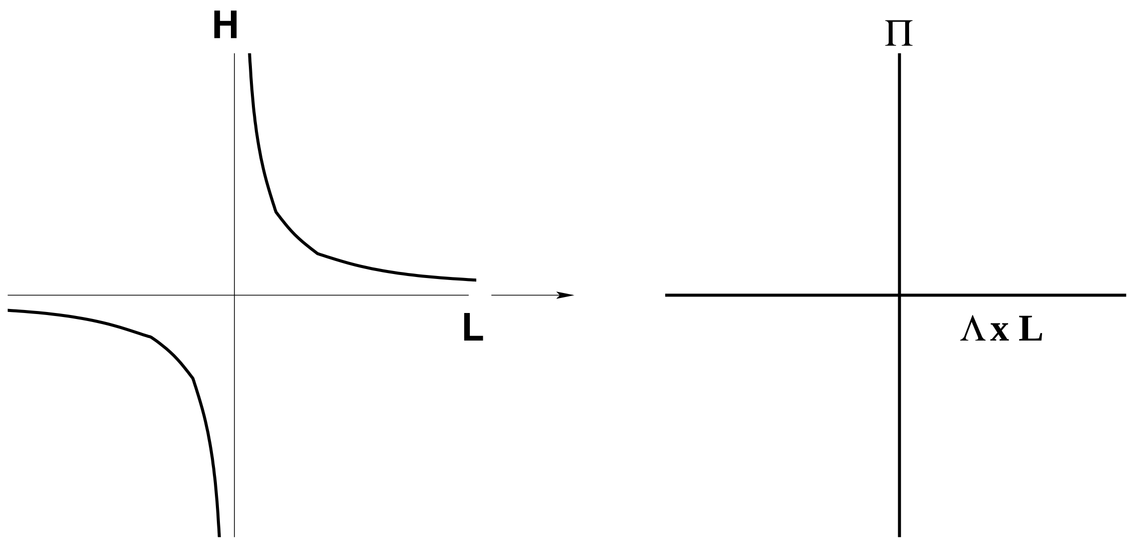
Vertex decompositions of Δ to Gröbner degenerations of X





Example: (a hyperbola) Let $X = \{(x, y) : xy = 1\}$, $H = x$-axis and $L = y$-axis.
The $\mathbb{k}^\star$ action deforms X to the variety X' cut out by $xy = t$. The projection piece is the y-axis. The "link" is $\Lambda = \{(0, 0)\}$ and the cone piece is the x-axis.
X' is reduced, a geometric vertex decomposition:

$$X' = \{(x, y) : xy = 0\} = \{x\text{-axis}\} \cup_{(0,0)} \{y\text{-axis}\}.$$



Example: $X = \mathrm{Spec}(\Bbbk[\mathbf{x}]/I_\Delta)$, L is the coordinate line corresponding to a vertex v and H is the hyperplane defined by the sum of the other coordinates.
$X = X'$ and the geometric vertex decomposition has each piece being the Stanley-Reisnerization of the vertex decomposition at v.

Geometric vertex decompositions can be useful to deduce Hilbert series or Gröbner bases of the projection/cone/link ideals from that of the original scheme.

Question: What further Gröbner geometry ideas can be inspired by basic operations on simplicial complexes?

Matrix Schubert varieties I

These generalized determinantal ideals are closely related to Schubert varieties in $GL_n(\mathbb{C})/B$ (also, see Alex Woo's tutorial talk later this week).

Let $\pi \in S_n$ be a permutation, thought of as a permutation matrix. Define the corresponding **rank matrix** r_π by

$$\pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to r_\pi = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

where $r_\pi(i,j) =$ number of 1's in the NW $i \times j$ matrix

Matrix Schubert varieties II

Let $Z = (z_{ij})$ be the coordinates of an $n \times n$ matrix and let Z_{ij} be the northwest $i \times j$ submatrix of Z.

Let I_π be the ideal of $\Bbbk[Z]$ generated by all $r_\pi(i,j)+1$ minors of Z_{ij}. Then the **matrix Schubert variety** is

$$X_\pi = \mathrm{Spec}(\Bbbk[Z]/I_\pi).$$

$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle .$$

Notice not all rank conditions/minors are **essential.**

<u>Exercise:</u> Which rank conditions/minors are essential?

Vertex decomposing matrix Schubert varieties

Example: Let us vertex decompose I_{2143} at the variable z_{33}. The initial z_{33}-form ideal is

$$\left\langle z_{11}, z_{33} \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \right\rangle$$

(Here were asserting that the defining minors are Gröbner enough for a one-step degeneration.)

This limit is reduced and has two (irreducible) components: The projection piece is $X_{2134} \cap (z_{33} = 0)$ and the cone piece is the line corresponding to z_{33} cross X_{3412}.

Proof:Look at the defining equations of these matrix Schubert varieties.

Macaulay 2 as a procedural language I

We want a procedure to generate all minors given an $n \times n$ rank matrix

```
generateminors = (G, Rank, n) -> (
  ans=trim(sum(flatten(for i from 0 to n-1 list
       for j from 0 to n-1 list minors(Rank_(n-i-1,j)+1,
          submatrix(G, {(n-i-1)..n-1}, {0..j})))));
  ans
);
```

Suppose we put this into a text file `file.m2`.

```
i1 : load "[YOUR M2 DIRECTORY]/file.m2";
--loaded [YOUR M2 DIRECTORY]/file.m2
```

Macaulay 2 as a procedural language II

```
i2 : R=QQ[x11,x12,x13,x14,x21,x22,x23,x24,x31,x32,x33,x34,x41,
x42,x43,x44,MonomialOrder=>Lex]
o2 = R
o2 : PolynomialRing
i3 : rankmatrix=matrix({{0,1,1,1},{1,2,2,2},
{1,2,2,3},{1,2,3,4}});
i4 : G=matrix({{x11,x12,x13,x14},{x21,x22,x23,x24},
{x31,x32,x33,x34},{x41,x42,x43,x44}});
i5 : generateminors(G,rankmatrix,4)
o5 = ideal (x41, x31, x21, x11,
            x33*x44 - x34*x43, [..snip..])
o5 : Ideal of R
```

Symmetric polynomials I

S_n has a natural action on $\mathbb{Z}[x_1, \ldots, x_n]$.

$$\pi \cdot f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(x_n)}).$$

f is **symmetric** if $\pi \cdot f = f$ for all $\pi \in S_n$.

Let $\Lambda[x_1, \ldots, x_n]$ denote the **ring of symmetric polynomials** in $\mathbb{Z}[x_1, \ldots, x_n]$.

There is the **Schur basis** $\{s_\lambda(\mathbf{x})\} \subseteq \Lambda[\mathbf{x}]$, indexed by partitions λ.

$$s_\lambda(\mathbf{x}) = \sum \mathbf{x}^T,$$

a sum over semistandard Young tableau of shape λ.

$$\lambda = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} : s_\lambda(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1 x_2 x_3^2 + x_2^2 x_3^2 + \ldots$$

Symmetric polynomials II

Consider an extension of semistandard tableau to **set-valued tableau**:

$$T = \begin{array}{|c|c|c|}\hline 12 & 23 & 45 \\ \hline 34 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \leftrightarrow (-1)^{\#\text{entries}-|\lambda|}\mathbf{x}^T = (-1)^{10-6}x_1^2x_2^2x_3^2x_4^2x_5^2x_6$$

$$G_\lambda(\mathbf{x}) = \sum (-1)^{\#\text{entries}-|\lambda|}\mathbf{x}^T$$

Observe: $G_\lambda(\mathbf{x}) = s_\lambda(\mathbf{x}) \pm$ higher degree stuff

Fact: $K(\Bbbk[\mathbf{x}]/I_1, \mathbf{1}-\mathbf{x}) = G_{2,2}(\mathbf{x})$.

The lowest homogeneous component of $K(\Bbbk[\mathbf{x}]/I_1, \mathbf{1}-\mathbf{x})$, i.e., $s_{2,2}(\mathbf{x})$ is the **multidegree**.
A goal of afternoon tutorial: Explore generalizations.

Schubert polynomials

We need a class of generalized Schur polynomials to be used this afternoon.
Define $\mathfrak{S}_{w_0}(\mathbf{x}) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ where $w_0 = n\ n-1\ \ldots 2\ 1$ is the longest word in S_n.

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}},$$

where $s_i = (i\ i+1)$ is a simple reflection.

$$\mathfrak{S}_\pi(\mathbf{x}) = \partial_i \mathfrak{S}_{ws_i}$$

if $\ell(ws_i) = \ell(w) + 1$ and ℓ is the **Coxeter length.**

<u>Theorem:</u> $\mathfrak{S}_\pi$ is a nonnegative integer sum of monomials.

Similarly...
Define $\mathfrak{G}_{w_0}(\mathbf{x}) = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ where $w_0 = n\ n-1\ \ldots 2\ 1$ is the longest word in S_n.

$$\Pi_i(f) = \frac{(1-x_{i+1})f - (1-x_i)s_i f}{x_i - x_{i+1}},$$

where $s_i = (i\ i+1)$ is a simple reflection.

$$\mathfrak{G}_\pi(\mathbf{x}) = \Pi_i \mathfrak{G}_{ws_i}$$

if $\ell(ws_i) = \ell(w) + 1$ and ℓ is the **Coxeter length.**

Theorem: $\mathfrak{G}_\pi = \mathfrak{S}_\pi \pm \cdots$ is an "alternating by degree" integer sum of monomials.

Comments on Schubert/Grothendieck polynomials

These polynomials were originally introduced to give polynomial representatives of Schubert classes in $H^\star(GL_n(\mathbb{C})/B, \mathbb{Z})$ and $K(GL_n(\mathbb{C})/B, \mathbb{Z})$.

They also represent the class of loci of vector bundles over a smooth projective space, with given degeneracy conditions.

<u>Lesson learned thus far:</u> $\mathfrak{G}_\pi$ should equal $K(X_\pi, \mathbf{1} - \mathbf{x})$ and $\mathfrak{S}_\pi$ should be the multidegree.

<u>Thematic question:</u> Gröbner degenerating matrix Schubert varieties reflects what combinatorics of Schubert polynomials?

In this lecture, we've introduced

- some theory in (combinatorially) studying ideals via Gröbner degeneration
- basics of `Macaulay 2` usage
- matrix Schubert varieties and hinted at various relations to symmetric and Schubert polynomial theory

The exercises/problems/projects given in the afternoon session will seek to further develop these ideas, e.g., what happens when you Gröbner degenerate arbitrary matrix Schubert varieties?

Some references I

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On matrix Schubert varieties and Schubert polynomials:

A. Knutson and E. Miller, *Gröbner geometry of Schubert polynomials*, Annals of Mathematics **161** (2005), 1245-1318.

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Thank You!