Scaling and Universality in Random Matrix Models

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Plan of the lectures

Lecture 1. General introduction to random matrix models.

Lecture 2. The Riemann-Hilbert approach to the large N asymptotics of orthogonal polynomials and random matrix models. Scaling limits and universality in the bulk of the spectrum and at the end-points.

Lecture 3. Double scaling limits and universality at critical points.

Lecture 4. Large N asymptotics of the free energy of random matrix models.

Lecture 1. General introduction to random matrix models

- **Unitary Ensemble of Random Matrices**

Let $M = \left(M_{jk}\right)_{j,k=1}^{N}$ be a random Hermitian matrix, $M_{kj} = \overline{M_{jk}}$, with respect to the probability distribution

$$\mu_N(dM) = Z_N^{-1} e^{-N \mathrm{Tr}\; V(M)} dM, \qquad M = M^{\dagger}.$$

where

$$V(M) = \sum_{i=1}^{p} t_j M^j, \qquad p = 2p_0, \qquad t_p > 0,$$

is a polynomial,

$$dM = \prod_{j=1}^{N} dM_{jj} \prod_{j \neq k}^{N} d\Re M_{jk} d\Im M_{jk},$$

the Lebesgue measure, and

$$Z_N = \int_{\mathcal{H}_N} e^{-N \mathrm{Tr}\; V(M)} dM,$$

the partition function.

- **Gaussian Unitary Ensemble (GUE)**

For $V(M) = M^2$,

$$\text{Tr } V(M) = \text{Tr } M^2 = \sum_{j,k=1}^{N} M_{kj} M_{jk}$$

$$= \sum_{j=1}^{N} M_{jj}^2 + 2 \sum_{j>k} |M_{jk}|^2,$$

hence

$$\mu_N(dM) = Z_N^{-1} \prod_{j=1}^{N} \left(e^{-NM_{jj}^2} dM_{jj} \right)$$

$$\times \prod_{j>k} \left(e^{-2N|M_{jk}|^2} d\Re M_{jk} d\Im M_{jk} \right),$$

so that the matrix elements are independent Gaussian random variables. If $V(M)$ is not quadratic then the matrix elements are dependent.

- **Topological Large N Expansion**

Free energy

$$F_N = -N^{-2} \ln \frac{Z_N}{Z_N^0}$$

$$= -N^{-2} \ln \frac{\int_{\mathcal{H}_N} e^{-N \mathrm{Tr}\, (M^2 + t_3 M^3 + t_4 M^4 + \dots)} dM}{\int_{\mathcal{H}_N} e^{-N \mathrm{Tr}\, (M^2)} dM}$$

$$= -N^{-2} \ln \left\langle e^{-N \mathrm{Tr}\, (t_3 M^3 + t_4 M^4 + \dots)} \right\rangle$$

$$= -N^{-2} \ln \Big\langle 1 - N \mathrm{Tr}\, (t_3 M^3 + t_4 M^4 + \dots)$$

$$+ \frac{1}{2!} N^2 [\mathrm{Tr}\, (t_3 M^3 + t_4 M^4 + \dots)^2] + \dots \Big\rangle .$$

where

$$\langle f(M) \rangle = \frac{\int_{\mathcal{H}_N} f(M) e^{-N \mathrm{Tr}\, M^2} dM}{\int_{\mathcal{H}_N} e^{-N \mathrm{Tr}\, M^2} dM}$$

Topological expansion:

$$F \sim F_0 + N^{-2}F_1 + N^{-4}F_2 + \dots$$

Expansion over Feynman diagrams:

$$F_j = \sum_{m=(m_3,m_4,\dots)} f_{jm}t^m, \quad t = (t_3, t_4, \dots),$$

where f_{jm} is (up to an explicit factor) the number of Feynman diagrams with m vertices on a Riemannian surface of genus j. Thus, F is a generating function for f_{jm}. It is used to find asymptotics of f_{jm} as $m \to \infty$.

Some references to topological expansions

1. E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Planar diagrams*, Commun. Math. Phys. **59** (1978), 35-51. D. Bessis, C. Itzykson, and J.-B. Zuber, *Quantum field theory techniques in graphical enumeration*, Adv. Appl. Math. **1** (1980), 109-157.

2. P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *2D gravity and random matrices*, Physics Reports **254** (1995), 1-131, and references therein.

3. P. Di Francesco, *Matrix model combinatorics: applications to folding and coloring*. In: "Random Matrices and Their Applications", MSRI Publications **40**. Eds. P. Bleher and A. Its, Cambridge Univ. Press (2001), 111-170.

4. N.M. Ercolani and K.D.T-R McLaughlin. Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *Int. Math. Res. Not.* **14** (2003), 755–820.

- **Ensemble of Eigenvalues**

$$\mu_N(d\lambda) = \tilde{Z}_N^{-1} \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)} d\lambda,$$

where

$$\tilde{Z}_N = \int \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)} d\lambda,$$
$$d\lambda = d\lambda_1 \dots d\lambda_N.$$

Main Problem: Find asymptotics of the partition function and correlations between eigenvalues as $N \to \infty$.

Correlation Functions

The m-point correlation function is given as

$$K_{mN}(x_1, \ldots, x_m)$$
$$= \frac{N!}{(N-m)!} \int_{\mathbb{R}^{N-m}} p_N(x_1, \ldots, x_N) dx_{m+1} \ldots dx_N,$$

where

$$p_N(x_1, \ldots, x_N) = \widetilde{Z}_N^{-1} \prod_{j>k} (x_j - x_k)^2 \prod_{j=1}^{N} e^{-NV(x_j)}.$$

Determinantal formula for correlation functions

$$K_{mN}(x_1,\ldots,x_m) = \det\left(Q_N(x_k,x_l)\right)_{k,l=1}^m,$$

where

$$Q_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y)$$

and

$$\psi_n(x) = \frac{1}{h_n^{1/2}} P_n(x) e^{-NV(x)/2},$$

where $P_n(x) = x^n + a_{n-1}x^{n-1} + \ldots$ are monic orthogonal polynomials,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)e^{-NV(x)}dx = h_n\delta_{nm}.$$

Recurrence and differential equations for orthogonal polynomials

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x),$$
$$\gamma_n = \left(\frac{h_n}{h_{n-1}}\right)^{1/2} > 0\,, \quad \gamma_0 = 0.$$

or

$$x\psi_n(x) = \gamma_{n+1}\psi_{n+1}(x) + \beta_n\psi_n(x) + \gamma_n\psi_{n-1}(x).$$

Consider the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^1)$,

$$\mathcal{H} = \{f(x) = \sum_{j=0}^{\infty} f_j \psi_n(x)\}, \quad f = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix},$$

with the scalar product $(f, g) = \sum_{j=0}^{\infty} f_j \overline{g_j}$. Consider the matrix Q of the operator of multiplication by x, $f(x) \to x f(x)$ in the basis $\{\psi_n(x)\}$. Then Q is the symmetric tridiagonal Jacobi matrix,

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & \cdots \\ \gamma_1 & \beta_1 & \gamma_2 & \cdots \\ 0 & \gamma_2 & \beta_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Christoffel-Darboux Formula

Calculation:

$$(x-y)\sum_{n=0}^{N-1}\psi_n(x)\psi_n(y)=\sum_{n=0}^{N-1}\Big[\Big(\gamma_{n+1}\psi_{n+1}(x)$$
$$+\beta_n\psi_n(x)+\gamma_n\psi_{n-1}(x)\Big)\,\psi_n(y)$$
$$-\psi_n(x)\left(\gamma_{n+1}\psi_{n+1}(y)+\beta_n\psi_n(y)+\gamma_n\psi_{n-1}(y)\right)\Big]$$
$$=\gamma_N\left[\psi_N(x)\psi_{N-1}(y)-\psi_{N-1}(x)\psi_N(y)\right]$$

(telescopic sum), hence

$$Q_N(x,y)=\sum_{n=0}^{N-1}\psi_n(x)\psi_n(y)$$
$$=\gamma_N\frac{\psi_N(x)\psi_{N-1}(y)-\psi_{N-1}(x)\psi_N(y)}{x-y}.$$

Density function:

$$p_N(x)=\frac{Q_N(x,x)}{N}$$
$$=\frac{\gamma_N}{N}\left[\psi'_N(x)\psi_{N-1}(x)-\psi'_{N-1}(x)\psi_N(x)\right].$$

Our goal is to derive semiclassical asymptotics for $\psi_n(z)$ on the complex plane, as $n, N \to \infty$ in such a way that

$$\frac{n}{N} \to \lambda > 0$$

(for Christoffel-Darboux we need $n = N, N-1$). There are three basic elements in the derivation:

1. String equations.

2. Lax pair equations.

3. Riemann-Hilbert problem.

- **String Equations**

Let $P = (P_{nm})_{n,m=0,1,2,\dots}$ be a matrix of the operator $f(z) \to f'(z)$ in the basis $\psi_n(z)$, $n = 0, 1, 2, \dots$. Then $P_{mn} = -P_{nm}$ and

$$\psi_n'(z) = -\frac{NV'(z)}{2}\psi_n(z) + \frac{P_n'(z)}{\sqrt{h_n}} e^{-NV(z)/2}$$
$$= -\frac{NV'(z)}{2}\psi_n(z) + \frac{n}{\gamma_n}\psi_{n-1}(z) + \dots,$$

hence

$$\left[P + \frac{NV'(Q)}{2}\right]_{nn} = 0,$$
$$\left[P + \frac{NV'(Q)}{2}\right]_{n,n+1} = 0,$$
$$\left[P + \frac{NV'(Q)}{2}\right]_{n,n-1} = \frac{n}{\gamma_n}.$$

Since $P_{nn} = 0$, we obtain that

$$[V'(Q)]_{nn} = 0. \qquad (*)$$

In addition,

$$0=\left[P+\frac{NV'(Q)}{2}\right]_{n-1,n}=\left[-P+\frac{NV'(Q)}{2}\right]_{n,n-1},$$
$$\left[P+\frac{NV'(Q)}{2}\right]_{n,n-1}=\frac{n}{\gamma_n},$$

hence

$$\gamma_n[V'(Q)]_{n,n-1}=\frac{n}{N}. \qquad (**)$$

Thus, we have the discrete string equations,

$$\begin{cases} [V'(Q)]_{nn}=0, \\ \\ \gamma_n[V'(Q)]_{n,n-1}=\frac{n}{N}. \end{cases}$$

Example. Quartic model,

$$V(M) = \frac{t}{2}M^2 + \frac{g}{4}M^4.$$

String equation,

$$\boxed{\gamma_n^2\left(t + g\gamma_{n-1}^2 + g\gamma_n^2 + g\gamma_{n+1}^2\right) = \frac{n}{N}}$$

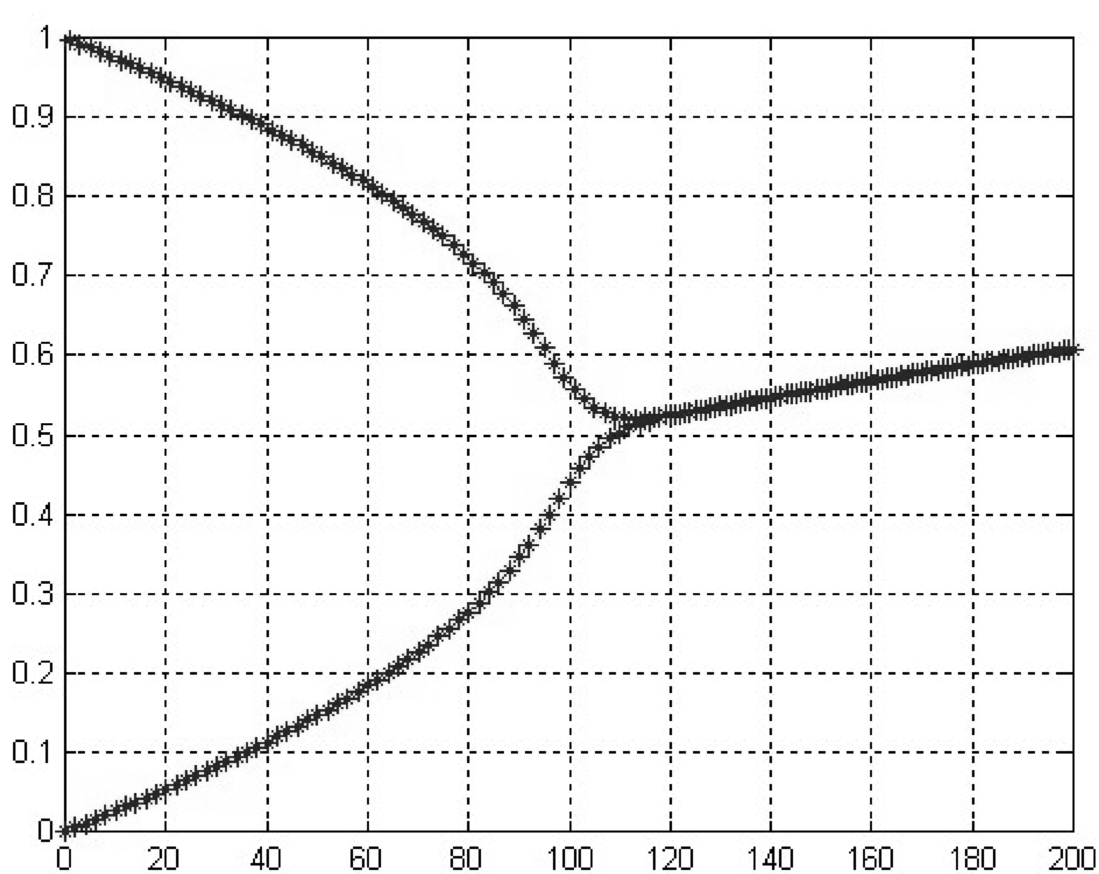
($\beta_n = 0$ and the second string equation is trivial in the case when $V(M)$ is even). Initial conditions: $\gamma_0 = 0$ and

$$\gamma_1 = \frac{\int_{-\infty}^{\infty} z^2 e^{-NV(z)}dz}{\int_{-\infty}^{\infty} e^{-NV(z)}dz}.$$

Gaussian model, $V(M) = \frac{M^2}{2}$, $t = 1$, $g = 0$:

$$\gamma_n^2 = \frac{n}{N}.$$

• **Computer solution of the string equation for the quartic model:** $g = 1$, $t = -1$, $N = 400$



- **Fix-point solution of the string equation:**

$$\gamma_n^2 = R\left(\frac{n}{N}\right),$$

$$R(\lambda) = \frac{-t + \sqrt{t^2 + 12g\lambda}}{6g}, \quad \lambda > \lambda_c = \frac{t^2}{2g}.$$

- **Period-2-solution of the string equation:**

$$\gamma_n^2 = \begin{cases} R\left(\frac{n}{N}\right), & n = 2k+1, \\ L\left(\frac{n}{N}\right), & n = 2k, \end{cases}$$

$$R(\lambda), L(\lambda) = \frac{-t \pm \sqrt{t^2 - 4g\lambda}}{2g}, \quad \lambda < \lambda_c.$$

- **Lax Pair Equations**

Define $\vec{\Psi}_n(z) = \begin{pmatrix} \psi_n(z) \\ \psi_{n-1}(z) \end{pmatrix}$.

Differential equation:

$$\vec{\Psi}'_n(z) = N A_n(z) \vec{\Psi}_n(z), \qquad (*)$$

where

$$A_n(z) = \begin{pmatrix} -\frac{V'(z)}{2} - \gamma_n u_n(z) & \gamma_n v_n(z) \\ -\gamma_n v_{n-1}(z) & \frac{V'(z)}{2} + \gamma_n u_n(z) \end{pmatrix}$$

and

$$u_n(z) = [W(Q,z)]_{n,n-1},$$
$$v_n(z) = [W(Q,z)]_{nn},$$

where

$$W(Q,z) = \frac{V'(Q) - V'(z)}{Q - z}.$$

Observe that $\mathrm{Tr}\ A_n(z) = 0$.

Recurrence equation:

$$\vec{\Psi}_{n+1}(z) = U_n(z)\vec{\Psi}_n(z), \qquad (**)$$

where

$$U_n(z) = \begin{pmatrix} \gamma_{n+1}^{-1}(z-\beta_n) & -\gamma_{n+1}^{-1}\gamma_n \\ 1 & 0 \end{pmatrix}$$

Thus, we have two equations on $\vec{\Psi}_n(z)$,

$$\begin{cases} \vec{\Psi}'_n(z) = NA_n(z)\vec{\Psi}_n(z), \\ \\ \vec{\Psi}_{n+1}(z) = U_n(z)\vec{\Psi}_n(z). \end{cases}$$

The compatibility conditions of these two equations are the discrete string equations, so that this is a Lax pair for the discrete strinq equations.

Example. Quartic model,

$$V(M) = \frac{t}{2}M^2 + \frac{g}{4}M^4.$$

Matrix $A_n(z)$:

$$A_n(z) = \begin{pmatrix} -\left[\left(\frac{t}{2} + g\gamma_n^2\right) z + \frac{gz^3}{2}\right] & \gamma_n(gz^2 + \theta_n) \\ -\gamma_n(gz^2 + \theta_{n-1}) & \left(\frac{t}{2} + g\gamma_n^2\right) z + \frac{gz^3}{2} \end{pmatrix}$$

where

$$\theta_n = t + g\gamma_n^2 + g\gamma_{n+1}^2.$$

- **Riemann-Hilbert Problem**

Adjoint functions to $\psi_n(z)$,

$$\varphi_n(z) = e^{\frac{NV(z)}{2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{NV(u)}{2}} \psi_n(u)\, du}{z-u}, \quad z \in \mathbb{C}.$$

Proposition 1. *The vector-valued function* $\vec{\Phi}_n(z) = \begin{pmatrix} \varphi_n(z) \\ \varphi_{n-1}(z) \end{pmatrix}$ *satisfies the Lax pair equations,*

$$\begin{cases} \vec{\Phi}_n'(z) = NA_n(z)\vec{\Phi}_n(z), \\ \vec{\Phi}_{n+1}(z) = U_n(z)\vec{\Phi}_n(z). \end{cases}$$

Define

$$\varphi_{n\pm}(x) = \lim_{\substack{z\to x \\ \pm\Im z>0}} \varphi_n(z), \qquad -\infty < x < \infty.$$

Then

$$\boxed{\varphi_{n+}(x) = \varphi_{n-}(x) + \psi_n(x).}$$

Asymptotics of $\varphi_n(z)$ as $z \to \infty$, $z \in \mathbb{C}$:

$$\varphi_n(z) = e^{\frac{NV(z)}{2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{NV(u)}{2}} \psi_n(u) \left(\sum_{j=0}^{\infty} \frac{u^j}{z^{j+1}} \right) du$$
$$= e^{\frac{NV(z)}{2}} \left(\frac{h_n^{1/2}}{2\pi i} z^{-n-1} + O\left(z^{-n-2}\right) \right)$$

(due to the orthogonality, the first n terms cancel out).

Psi-matrix:

$$\Psi_n(z) = \begin{pmatrix} \psi_n(z) & \varphi_n(z) \\ \psi_{n-1}(z) & \varphi_{n-1}(z) \end{pmatrix}$$

Lax pair:

$$\begin{cases} \Psi'_n(z) = NA_n(z)\Psi_n(z), \\ \Psi_{n+1}(z) = U_n(z)\Psi_n(z) \end{cases}$$

WKB asymptotic solution:

$$\Psi_n(z) = V_n(z)e^{N\Lambda_n(z)}$$

where $\Lambda_n(z) = \text{diag}\ (\lambda_{n1}(z), \lambda_{n2}(z))$. Then

$$\Lambda'_n = V_n^{-1}A_nV_n - N^{-1}V_n^{-1}V'_n$$

In the leading order, $\Lambda'_n = V_n^{-1}A_nV_n$, so that $\lambda'_{n1}, \lambda'_{n2}$ are eigenvalues of A_n, and V_n is the matrix of eigenvectors of A_n. Since $\text{Tr}\ A_n = 0$,

$$\Psi_n(z) = V_n(z)e^{N\lambda_n(z)\sigma_3},$$

where $\lambda'_n(z) = \sqrt{-\det A_n(z)}$.

Riemann-Hilbert problem for $\Psi_n(z)$:

- $\Psi_n(z)$ is analytic on $\{\Im z \geq 0\}$ and $\{\Im z \leq 0\}$ (two-valued on $\{\Im z = 0\}$).

- $\Psi_{n+}(z) = \Psi_{n-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Im z = 0.$

- $\Psi_n(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k} \right) e^{-(NV(z)/2 - n \ln z + \lambda_n)\sigma_3}$, $z \to \infty$, where Γ_k, $k = 0, 1, 2, \ldots$, are some constant 2×2 matrices, with

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & c_n \end{pmatrix},$$

where λ_n and $c_n \neq 0$ are some explicit constants, and σ_3 is the Pauli matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

• Riemann-Hilbert Problem for Orthogonal Polynomials

- $Y_n(z)$ is analytic on $\{\Im z \geq 0\}$ and $\{\Im z \leq 0\}$ (two-valued on $\{\Im z = 0\}$).

- For any real x,

$$Y_{n+}(x) = Y_{n-}(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix},$$

  where $w(x) = e^{-NV(x)}$.

- As $z \to \infty$,

$$Y_n(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{Y_k}{z^k} \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

  where Y_k, $k = 1, 2, \ldots$, are some constant 2×2 matrices.

The RH problem has a unique solution

$$Y_n(z) = \begin{pmatrix} P_n(z) & C(wP_n)(z) \\ c_n P_{n-1}(z) & c_n C(wP_{n-1})(z), \end{pmatrix}$$

where

$$C(wP_n)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{w(x) P_n(x) dx}{x - z},$$

and $c_n = -2\pi i (\gamma_{n-1})^2$. The recurrent coefficients can be found as

$$\gamma_n^2 = [Y_1]_{21} [Y_1]_{12},$$

$$\beta_n = \frac{[Y_2]_{21}}{[Y_1]_{21}} - [Y_1]_{11}.$$

We will construct a semiclassical solution (parametrix) to the RH problem in several steps. The first step is based on the equilibrium measure for the function $V(x)$.

• Distribution of Eigenvalues and Equilibrium Measure

Rewrite the distribution of eigenvalues

$$d\mu_N(\lambda) = Z_N^{-1} \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)} d\lambda_j,$$

as $d\mu_N(\lambda) = Z_N^{-1} e^{-H_N(\lambda)} d\lambda$ where

$$H_N(\lambda) = -\sum_{j \neq k} \log|\lambda_j - \lambda_k| + N \sum_{j=1}^{N} V(\lambda_j)$$
$$= N^2 \left[-\iint_{x \neq y} \log|x-y| d\nu_\lambda(x) d\nu_\lambda(y) \right.$$
$$\left. + \int V(x) d\nu_\lambda(x) \right] \equiv N^2 I_V(\nu_\lambda)$$

and $d\nu_\lambda(x) = N^{-1} \sum_{j=1}^{N} \delta(x - \lambda_j) dx$.

Thus,

$$d\mu_N(\lambda) = Z_N^{-1} e^{-N^2 I_V(\nu_\lambda)} d\lambda.$$

We expect that for large N the measure $d\mu_N(\lambda)$ is concentrated near the minimum of the functional I_V, i.e. near the equilibrium measure $d\nu(x)$.

- **Equilibrium Measure**

Consider the minimization problem

$$E_V = \inf_{\nu \in M_1(\mathbb{R})} I_V(\nu),$$

where

$$M_1(\mathbb{R}) = \left\{ \nu \, : \, \int_{\mathbb{R}} d\nu = 1 \right\}$$

and

$$I_V(\nu) = -\iint \log|s-t| d\nu(s) d\nu(t) + \int V(t) d\nu(t).$$

Proposition 2.2. *The infinum of $I_V(\nu)$ is achieved uniquely at an equilibrium measure $\nu = \nu_V$. The measure ν_V is supported by a finite union of intervals, $J = \cup_{j=1}^{q}[a_j, b_j]$, and on J it has the form*

$$d\nu(x) = p(x)dx,$$

where

$$p(x) = \frac{1}{2\pi i} h(x) R_+^{1/2}(x),$$
$$R(x) = \prod_{j=1}^{q} (x - a_j)(x - b_j).$$

Here $R^{1/2}(x)$ is the branch with cuts on J, which is positive for large positive x and $R_+^{1/2}(x)$ is the value of $R^{1/2}(x)$ on the upper part of the cut. The function $h(x)$ is a polynomial, which is the polynomial part of the function $\frac{V'(x)}{R^{1/2}(x)}$ at infinity, i.e.

$$\frac{V'(x)}{R^{1/2}(x)} = h(x) + O(x^{-1}).$$

In particular, $\deg h = \deg V - 1 - q$.

- **A useful formula for the equilibrium density**

$$\frac{d\nu_V(x)}{dx} = \frac{1}{\pi}\sqrt{q(x)},$$

where

$$q(x) = \left(\frac{V'(x)}{2}\right)^2 - \int \frac{V'(x) - V'(y)}{x - y} d\nu_V(y).$$

Reference

P. Deift, T. Kriecherbauer, and K.T-R McLaughlin. New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *J. Approx. Theory* **95** (1998), 388–475.

The Euler-Lagrange variational conditions: for some real constant l,

$$2\int \log|x-y|d\nu(y) - V(x) = l, \text{ for } x \in J,$$

$$2\int \log|x-y|d\nu(y) - V(x) \le l, \text{ for } x \in \mathbb{R}\setminus J$$

Definition. *The equilibrium measure*

$$\nu(dx) = \frac{1}{\pi i} h(x) R_+^{1/2}(x)\, dx$$

is regular (otherwise singular) if

1. $h(x) \neq 0$ *on the (closed) set* J,

2. *The inequality is strict,*

$$2\int \log|x-y|d\nu(y) - V(x) < l, \text{ for } x \in \mathbb{R}\setminus J.$$

Example. If $V(x)$ is convex then $\nu(dx)$ is regular and the support of $\nu(dx)$ consists of a single interval.

- **Equations on the End-Points**

Define

$$\omega(z) = \int_J \frac{\rho(x)dx}{z-x}, \quad z \in \mathbb{C} \setminus J.$$

where $d\mu(x) = \rho(x)dx$ is the equilibrium measure. The Euler-Lagrange variational condition implies that

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}.$$

Observe that as $z \to \infty$,

$$\omega(z) = \frac{1}{z} + \frac{m_1}{z^2} + \dots, \quad m_k = \int_J x^k \rho(x)dx.$$

The equation

$$\frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2} = \frac{1}{z} + O(z^{-2}).$$

gives $q+1$ equations on $a_1, b_1, \dots, a_q, b_q$. Remaining $q-1$ equations are

$$\int_{b_j}^{a_{j+1}} h(x)R^{1/2}(x)\,dx = 0, \quad j = 1, \dots, q-1.$$

Example. Quartic model,

$$V(M) = \frac{t}{2}M^2 + \frac{1}{4}M^4.$$

For $t \geq t_c = -2$, the support of the equilibrium distribution consists of one interval $[-a, a]$ where

$$a = \left(\frac{-2t + 2\left(t^2 + 12\right)^{1/2}}{3} \right)^{1/2}$$

and

$$\frac{d\nu_V(x)}{dx} = \frac{1}{\pi}\left(b + \frac{1}{2}x^2\right)\sqrt{a^2 - x^2}$$

where

$$b = \frac{t + \left((t^2/4) + 3\right)^{1/2}}{3}.$$

In particular, for $t = -2$,

$$\frac{d\nu_V(x)}{dx} = \frac{1}{2\pi}x^2\sqrt{4 - x^2}$$

For $t < -2$, the support consists of two intervals, $[-a, -b]$ and $[b, a]$, where

$$a = \sqrt{2 - t}, \quad b = \sqrt{-2 - t}\,,$$

and

$$\frac{d\nu_V(x)}{dx} = \frac{1}{2\pi}|x|\sqrt{(a^2 - x^2)(x^2 - b^2)}\,.$$

- **The density function for $t = -1, -2, -3$.**