"How Accurate is KdV Approximation to the Water Wave Equation?"

Fields Seminar
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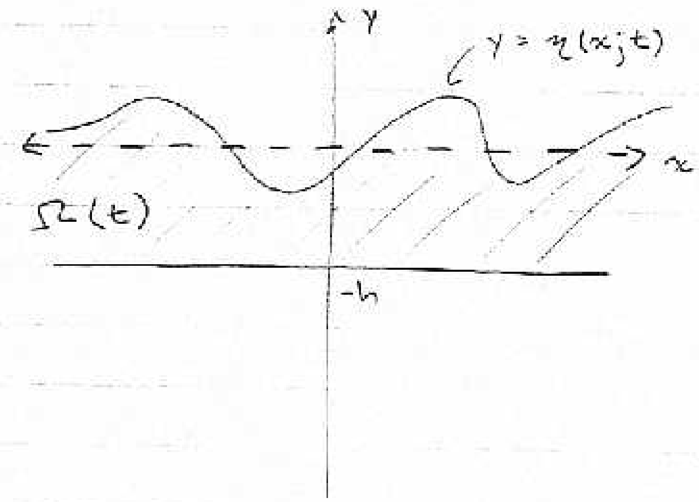
- Fields & McMaster

The Water Wave Equation As we saw it last week:



If the free surface is a graph over the x-coordinate, i.e.

$\Gamma(t)$ = free surface

$= \{(x, \eta(x;t)) \mid x \in \mathbb{R},\ t \geq 0\}$

And if we denote the fluid velocity at position (x,y) at time t by $\underline{u}(x,y;t)$, then the evolution of the surface is given by:

(u_1, u_2)

$$
WE \begin{cases}
\left.\begin{array}{l} \nabla \cdot \underline{u} = 0 \\ \nabla \times \underline{u} = 0 \end{array}\right\} \text{ in } \Omega(t) & \leftarrow \text{incompressible, irrotational} \\[1ex]
u_2(x,-h) = 0 & \leftarrow \text{no flow through bottom} \\[1ex]
\partial_t \eta = u_2 - u_1 \partial_x \eta \quad \text{on free surface} & \leftarrow \text{The kinematic Boundary Condition} \\[1ex]
\partial_t \underline{u} + (\underline{u} \cdot \nabla)\underline{u} = -\nabla p - \begin{pmatrix} 0 \\ g \end{pmatrix} \quad \text{on free surface} & \leftarrow \text{Euler's Eqn}
\end{cases}
$$

The "Eulerian" Formulation

We can reformulate this as follows. Let us track the position of fluid particles, instead of having a fixed lab frame.

Step 1 Parameterize the free surface as follows

$$\Gamma(t) = \{ (\tilde{X}(\alpha;t), Y(\alpha;t)) \mid \alpha \in \mathbb{R},\ t \geq 0 \}$$

Here $\underbrace{(\tilde{X}(\alpha;t), Y(\alpha;t))}_{\underline{X}(\alpha,t)}$ = Position at time t of the "particle" which was at $(\tilde{X}(\alpha;0), Y(\alpha;0))$ at time $t=0$.

We will find equations which give the evolution of $\tilde{X}$ & Y.

The key change of variables is:

$$\partial_t \underline{X}(\alpha;t) = \underline{u}(\underline{X}(\alpha;t), t)$$

on the surface

So: $$\partial_t^2 \underline{X} = \partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla}) \underline{u}$$

From (W.E.)

$$= -\nabla p - \begin{pmatrix} 0 \\ g \end{pmatrix}$$

or rather

$$\boxed{\partial_t^2 \underline{X}(\alpha;t) = -\nabla p\,(\underline{X}(\alpha,t),t) - \begin{pmatrix} 0 \\ g \end{pmatrix}} \quad (\star)$$

Now the pressure, p, is constant on the free surface, i.e.

$$p(\underline{X}(\alpha,t),t) = \text{constant}$$

$$\partial_\alpha \big(p(\underline{X}(\alpha,t),t) \big) = 0$$

$$\nabla p(\underline{X}(\alpha,t),t) \cdot \big(\partial_\alpha \tilde{X}(\alpha,t), \partial_\alpha Y(\alpha,t)\big) = 0$$

So we remove ∇p from ⊛ by taking the inner product of ⊛ w/ $(\partial_\alpha \tilde{X}, \partial_\alpha Y)$ and we have

$$(\partial_t^2 \tilde{X}, \partial_t^2 Y) \cdot (\partial_\alpha \tilde{X}, \partial_\alpha Y) = -(0,g) \cdot (\partial_\alpha \tilde{X}, \partial_\alpha Y)$$

$$\partial_t^2 \tilde{X}\, \partial_\alpha \tilde{X} + \partial_t^2 Y \partial_\alpha Y = -g\, \partial_\alpha Y$$

I am lazy, so $g = 1$. For reasons technical, we take

$$\tilde{X}(\alpha;t) = \alpha + X(\alpha;t)$$

we have

$$\boxed{\partial_t^2 X\,(1 + \partial_\alpha X) + \partial_\alpha Y\,(1 + \partial_t^2 Y) = 0}$$

Rmk 1. the kinematic bndry condition reduces in a similar fashion to

$\partial_t Y = u_2$ ← which is boring

Now notice

$$\partial_x u_1 + \partial_y u_2 = 0$$
$$\partial_x u_2 - \partial_y u_1 = 0$$

Are almost C-R eqns (- sign problem)

the Complex function $f(x+iy) = u_1(x,y) - i u_2(x,y)$
is analytic in $\Omega(t)$.

Def'n: Given a region Ω in $\mathbb{C}$, with boundary $\delta\Omega$ there is a (linear) relationship between the real and imaginery components of analytic function on Ω when restricted to $\delta\Omega$. That is, $\exists$ $H(\Omega)$ linear such that if $f(z) = u(x,y) + i v(x,y)$

$$v|_{\delta\Omega} = H(\Omega)\, u|_{\delta\Omega}$$

Called the "Hilbert Transform of Ω"

Sol

$$u_2 = H(\Omega(t))(-u_1)$$

and recalling:

$$u_2(x, \eta(x,t)) = \partial_t Y(\alpha;t)$$
$$u_1(x, \eta(x,t)) = \partial_t X(\alpha;t)$$

we have

$$\partial_t Y = -H(\Omega(t))\, \partial_t X$$

Note: $\Omega(t)$ is determined by (X,Y), so we let

$$K\big(X(\alpha;t), Y(\alpha;t)\big) = -H(\Omega(t))$$

Eg| If Ω = lower half plane,

then $H(\Omega)$ = most typical Hilbert transform H

where $\widehat{Hf}(k) = i\,\mathrm{sgn}(k)\,\hat{f}(k)$ ⟵ hats go on Fourier Transform

Eg| If $\Omega_0 = \{(x,y) \mid x \in \mathbb{R},\ -1 \le y \le 0\}$ ⟵ unit strip

and if $f(z)$ is analytic in Ω_0 and $\operatorname{Im} f \big|_{(x,-1)} = 0$

then

$$\widehat{H(\Omega_0)u}(k) = i\tanh(k)\,\hat{u}(k)$$

Ck: $f = u + iv$ analytic $\Rightarrow$ $v_{xx} + v_{yy} = 0$ in Ω_0

and $\partial_x u = \partial_y v$

Take F.T. wrt x coord

$$\rightsquigarrow \quad -k^2\hat{v} + \partial_y^2 \hat{v} = 0 \Rightarrow \partial_y^2 \hat{v}(k,y) = +k^2 \hat{v}(k,y) \quad ; \quad \hat{v}(k,-1) = 0$$

$$\hat{v}(k,y) = A(k)\sinh(k(y+1))$$

$$ik\,\hat{u}(k,y) = \partial_y \hat{v}(k,y) = A(k)\,k\cosh(k(y+1))$$

$$\rightsquigarrow \hat{u}(k,y) = -iA(k)\cosh(k(y+1))$$

$$\rightsquigarrow \frac{\hat{v}(k,y)}{\hat{u}(k,y)} = i\tanh(k(y+1))$$

$$\rightsquigarrow \boxed{\hat{v}(k,0) = i\tanh(k)\,\hat{u}(k,0)}$$

If $h = -1$, then (under certain conditions)

$$K(X,Y)u = K_0 u + K_1(X,Y)u + S_2(X,Y)u$$

Thank you W. Craig!!

where $\widehat{K_0 u} = \underbrace{-i\tanh(k)}_{\hat{K}_0(k)}\hat{u}(k)$

$K_1(X,Y)u = [X, K_0]\partial_x u - (Y + K_0 Y K_0)\partial_x u$

& $S_2(X,Y)$ = Quadratic in X, Y

Sol

$$\boxed{\begin{aligned} \partial_t^2 X(1+\partial_\alpha X) + \partial_\alpha Y(1+\partial_t^2 Y) &= 0 \quad (1) \\ \partial_t Y = K(X,Y)\,\partial_t X & \quad (2) \end{aligned}}$$

WW

There are reasons why this formulation is better. For our purpose its better because Thms can be proven in this setting that are hard to prove in WE.

Q: Hey, where's KdV / Bouss ?

More Brute force than we saw last week. (Burstill Formal)

Assume long waves, small amplitudes, slowly varying solns

⇝ take $0 < \varepsilon < 1$

$\beta = \varepsilon\alpha, \quad \tau = \varepsilon t$

$$X(\alpha,t) = \varepsilon \mathcal{X}(\beta,\tau) + O(\varepsilon^3)$$
$$Y(\alpha,t) = \varepsilon^2 Y_1(\beta,\tau) + \varepsilon^4 Y_2(\beta,\tau) + O(\varepsilon^6)$$

Does this scaling mystify you?

Notice ~ $\partial_t \rightsquigarrow \varepsilon\partial_\tau \qquad \partial_\alpha \rightsquigarrow \varepsilon\partial_\beta$

Also Notice we can expand $\hat{K}_0(k)$ in T.S.

$$-i\tanh(k) = -ik - \tfrac{1}{3}(ik)^3 - \tfrac{2}{15}(ik)^5 + \mathcal{O}(k^5)$$

& knowing under F.T. that $\partial_\alpha \longleftrightarrow ik$

we see

$$K_0 = -\partial_\alpha - \tfrac{1}{3}\partial_\alpha^3 + \mathcal{O}(\partial_\alpha^5)$$

or in long wave limit

$$K_0 \sim -\varepsilon\partial_\beta - \tfrac{\varepsilon^3}{3}\partial_\beta^3 + \mathcal{O}(\varepsilon^5)$$

Prop $\|K_0 f + \partial_\alpha f\|_s \leq C\|\partial_\alpha^3 f\|_s$ if $f \in H^{s+3}$, $s \geq 0$

$$\int_{-\infty}^{\infty} (1+k^2)^s \left|(\hat{K}_0(k) + ik)\hat{f}(k)\right|^2 dk$$

$$\leq \int_{-\infty}^{\infty} (1+k^2)^s \, C|k|^6 |\hat{f}|^2(k)\, dk$$

$$\leq C\|\partial_\alpha^3 f\|_s$$

Cor if $f(\alpha) = F(\varepsilon\alpha)$, then

$$\|K_0 f + \partial_\alpha f\|_s \leq C\varepsilon^{5/2}\|F\|_{s+3}$$

Rmk: We get similar estimates for higher approximations of K_0. We trade smoothness for smallness

Point: We can now make long wave approximations to everything in (WW)

Start:

$$\partial_t Y = K(X,Y)\,\partial_t X$$

$$= \left(K_0 + [X, K_0]\partial_\alpha - (Y + K_0(Y K_0))\partial_\alpha + S_2(\cdots)\right)\partial_t X$$

$$\partial_t Y = \varepsilon^3 \partial_\tau y_1 + \varepsilon^5 \partial_\tau y_2 + O(\varepsilon^7)$$

& $\quad \text{RHS} = \varepsilon^3(-\partial_\beta\partial_\tau x) + \varepsilon^5\left(-\frac{1}{3}\partial_\beta^3\partial_\tau x + \partial_\beta x\,\partial_\beta\partial_\tau x - y_1\partial_\beta\partial_\tau x\right)$

$\rightsquigarrow$ so

$$\left.\begin{aligned} \partial_\tau y_1 &= -\partial_\beta\partial_\tau x \\ \Rightarrow\quad y_1 &= -\partial_\beta x \end{aligned}\right\} \quad O(\varepsilon^3)$$

& sim at $O(\varepsilon^5)$

$$y_2 = -\frac{1}{3}\partial_\beta^3 x + (\partial_\beta x)^2$$

Now we have y_1, y_2 in terms of x, so sub everything into (WW1)

$$\partial_\tau^2\partial_\beta x = \partial_\beta^3 x + \varepsilon^2\left(\frac{1}{3}\partial_\beta^5 x - \frac{3}{2}\partial_\beta^2(\partial_\beta x)^2\right)$$

Boussinesq

If $z = -\partial_\beta x$ we get:

$$\partial_\tau^2 z - \partial_\beta^2 z = \varepsilon^2\left(\frac{1}{3}\partial_\beta^4 z + \frac{3}{2}\partial_\beta^2(z^2)\right)$$

To get to KdV, let

$$Z = A_r(\underbrace{\beta - \tau}_{\beta_-}, \underbrace{\varepsilon^2 \tau}_{T}) + A_\ell(\underbrace{\beta + \tau}_{\beta_+}, \varepsilon^2 \tau)$$

& we have: (to $O(\varepsilon^4)$)

$$\left.\begin{aligned} 2\partial_T A_r &= -\tfrac{1}{3}\partial_{\beta_-}^3 A_r - \tfrac{3}{2}\partial_{\beta_-}(A_r^2) \\ 2\partial_T A_\ell &= \tfrac{1}{3}\partial_{\beta_+}^2 A_\ell + \tfrac{3}{2}\partial_{\beta_+}(A_\ell^2) \end{aligned}\right\} \text{KdV}$$

Backing up

formally:

$$\boxed{\begin{aligned} -\partial_\alpha X(\alpha,t) &= \varepsilon^2\left(A_\ell(\varepsilon(\alpha+t), \varepsilon^3 t) + A_r(\varepsilon(\alpha-t), \varepsilon^3 t)\right) + O(\varepsilon^4) \\ Y(\alpha,t) &= \varepsilon^2\left(A_\ell(\varepsilon(\alpha+t), \varepsilon^3 t) + A_r(\varepsilon(\alpha-t), \varepsilon^3 t)\right) + O(\varepsilon^4) \end{aligned}}$$

where A_ℓ, A_r satisfy KdV eqns

How do we prove that this is true?

~~Step 1. Rewrite (WW) as an evolution equation in a "good" set of variables (preferably things will be quasilinear!)~~

Rmk 1 if One examines the linearized eqns, one finds that X (may) will grow linearly in time, but $\partial_\alpha X$ and $K_0 X \sim -\partial_\alpha X$ Do not so we will examine the W.W.E in the following variables

$$Z = K_0 X$$
$$U = \partial_t X$$
and Y

So we have seen formally $Z \sim \overbrace{\varepsilon^2 (A_l + A_r)}^{\varepsilon^2 \Psi^Z}$
$Y \sim \varepsilon^2 (A_l + A_r) \sim \varepsilon^2 \Psi^Y$
one can check that $U \sim \varepsilon^2 (A_r - A_l) \sim \varepsilon^2 \Psi^u$

So we want to mesure the difference between a true sol'n to WW & the approx Thus we let

$$Z^{(x,t)} = \varepsilon^2 \Psi^{Z(x,t)} + \varepsilon^{7/2} R^Z (x,t)$$
$$Y^{(x,t)} = \varepsilon^2 \Psi^Y + \varepsilon^{7/2} R^Y$$
$$U^{(x,t)} = \varepsilon^2 \Psi^u + \varepsilon^{7/2} R^u$$

This Analysis compliments of G. Schneider & C. Wayne

and we want to derive the evolution equations for (R^Z, R^Y, R^u) If we can do so in away which proves R^Z, R^Y, R^u remain $O(1)$ for some amount of time, we're okay!

THIS is Wicked Hard to do!

End result you have ε_0 st if $\varepsilon \in [0, \varepsilon_0]$

$$\| y(\cdot,t) - \varepsilon^2 \Psi^y(\cdot,t) \|_s \leq C \varepsilon^{7/2}$$

for $t \in [0, T_0 \varepsilon^{-1}]$. $C \neq C(\varepsilon, \varepsilon_0)$ & $C = C(T_0)$

$\varepsilon_0 = \varepsilon_0(T_0)$ as well

Sol What are the predictions?

① WW have a solitary wave

(A_ℓ = solitary wave
$A_r = 0$)

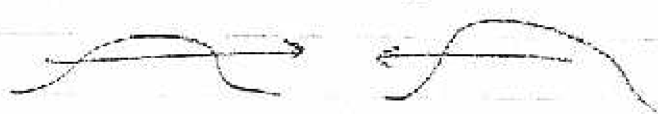
moves loosely w/ speed 1

② WW have a head on collision

A_ℓ = soliton , A_r = soliton

This is "fast" collision

$t \sim O(1)$

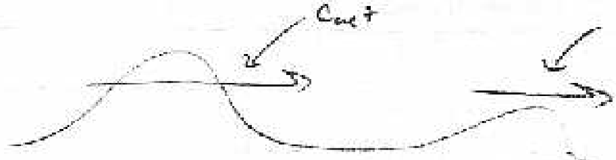


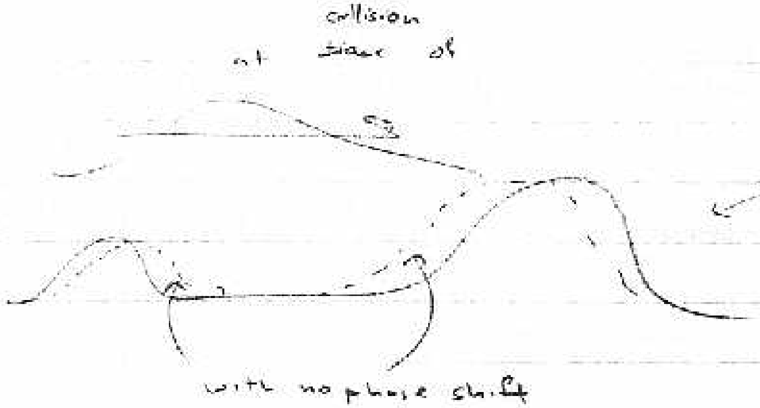
At collision, Waves add in Supper position

③ Things akin to Multiple Solitons exist in Water waves

fast

slow





$t \sim O(\varepsilon^{-3})$

Rmk: To see this sort of behavior T_0 may need to be large $T_0 \sim 10$, which may make C large & ε_0 small

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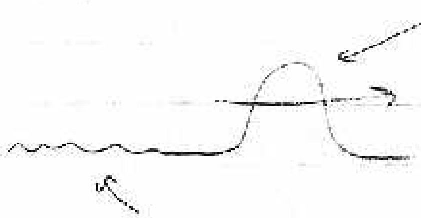
What isn't so good?

① Dispersive Waves



Problem is "unidirectional"

get a ripple → KdV doesn't see it

② At time of collision in a head on collision height is slightly more than linear superposition

from an interaction of left & right waves

(Also interaction here is "fast")

(3) Phase Shift isn't predicted very well

My Thesis ⇝ Fix it up!

How: Keep KdV, but add stuff ⟶ there are other ways!

Ansatz

same as before

"fudge factor"

$$y(\alpha,t) \cong \varepsilon^2\left(A_\ell(\varepsilon(\alpha+t),\varepsilon^3 t) + A_r(\varepsilon(\alpha-t),\varepsilon^3 t)\right) + \varepsilon^4 F(A_\ell, A_r)$$

$$+ \varepsilon^4\left(P(\varepsilon\alpha,\varepsilon t) + \underbrace{B_\ell(\varepsilon(\alpha+t),\varepsilon^3 t) + B_r(\varepsilon(\alpha-t),\varepsilon^3 t)}\right)$$

Fix interaction problem

Fix "unidirectional" problems

Call this RHS $\varepsilon^2\psi_2$

Grinding through the formal Computation, one has

W) $\partial_\tau^2 P - \partial_\beta^2 P = 3\partial_\beta^2\left(A_\ell(\ldots)A_r(\ldots)\right)$

$$\text{IV}\begin{cases} -2\partial_T B_r = \tfrac{1}{3}\partial_{\beta_-}^3 B_r + \tfrac{3}{2}\partial_{\beta_-}(A_r B_r) + J_r \\ 2\partial_T B_\ell = \tfrac{1}{3}\partial_{\beta_+}^3 B_\ell + \tfrac{3}{2}\partial_{\beta_+}(A_\ell B_\ell) + J_\ell \end{cases}$$

in hom. dep on A_r, P & their derivatives

(IW) is solvable easily & explicitly via (say) method of characteristics. If A_ℓ, A_r are "spatially localized" (like a soliton), then Ψ is Bounded over the long time scale

(LKdV) are also explicitly Solvable (R. Sachs, Sattinger & Haragus-Courcelle) Though this is a night-marish business. They are easy to compute numerically.

① Problem ⇝ Sol'ns to these equation grow with time $\rightarrow$ $\|A_\ell(\cdot,T)\| \leq CTe^{cT}$

② But who cares, $T = \varepsilon^3 t$ is $O(1)$ on the long time scale

Thm shows $\|y(\cdot,t) - \varepsilon^2 \Psi_2^Y(\cdot,t)\|_s \leq C\varepsilon^{1/2}$

for $t \in [0, T_0\varepsilon^{-3}]$

Pf: Same as before ...

Picture Time

Further ① Improves accuracy, how to improve time scale?
② Global in time sol'ns like multi-solitons?
(or really any "unidirectional" sol'ns?)