1. Ramsey theory and Banach spaces, Fields Institute, Fall 2002. Problem Set #3
October 21, 2002

Due date is **November 4, 2002**.

Exercise 1. *Assume S is a compact right-topological semigroup.*

(1) *If p is an idempotent in S then there is a minimal idempotent $q \leq p$.*
(2) *Assume S_1 is a closed subsemigroup of S such that $I = S \setminus S_1$ is an ideal, and p is a minimal idempotent in S_1. Assume q is an idempotent in I. Then for every continuous homomorphism $f: S \to S_1$ such that $f \restriction S_1$ is an identity we have $f(q) = p$.*

(Note: An example for the objects as in the previous exercise is $S = \bigcup_{i=1}^{k} \mathrm{FIN}^i$, $S_1 = \bigcup_{i=1}^{k-1} \mathrm{FIN}^i$, and $I = \mathrm{FIN}^k$. And the proof of (2) can be modified to show that $T(q) = p$, if T is the tetris operation. Hence this shows that the idempotents p_k in FIN^k can be chosen to be minimal, as I have originally stated.)

Exercise 2. *Prove that for $A \subseteq \mathbb{N}$ the following are equivalent:*

(1) *For every partition of $\mathbb{N}$ into finitely many pieces there is an infinite subset B of A such that $\mathrm{FS}(B)$ is included in a single piece.*
(2) *There is $p \in \beta\mathbb{N}$ such that $p + p = p$.*

(Notation: $\mathrm{FS}(A)$ stands for all finite sums of distinct elements of A.)

An ultrafilter p on FIN_k is *cofinite* if for every $m \in \mathbb{N}$ the set $\{s \in \mathrm{FIN}_k : \mathrm{supp}(s) \cap m = \emptyset\}$ belongs to p.

Exercise 3. *Prove that the cofinite ultrafilters on FIN_k form a closed subsemigroup of $\beta\,\mathrm{FIN}_k$.*

Let Σ be a finite alphabet and let S be the semigroup of words over Σ. In other words, S is the free semigroup with Σ as the set of generators. Let v be a variable which is not a member of Σ. A word over $\Sigma \cup \{v\}$ is a *variable word* if v occurs in it. Let $S(v)$ be the semigroup of all variable words. If $x \in S(v)$ and $a \in \Sigma$, let $x(a)$ be the word obtained by replacing all occurrences of v in x by a.

Theorem 4 (Hales–Jewett)**.** *If $S = \bigcup_{i=1}^{k} Y_i$, then there is a variable word x and $i \leq k$ such that $\{x(a) : a \in \Sigma\} \subseteq Y_i$.*

Exercise 5. *Prove Hales–Jewett theorem.* [1]

There is also an infinite version of Hales–Jewett theorem, that combines it with Hindman's theorem for the free semigroup.

Theorem 6. *If $S = \bigcup_{i=1}^{k} Y_i$, then there is a sequence of variable words $A = \langle w_i : i \in \mathbb{N}\rangle$ and $j \leq k$ such that the subspace generated by A is included in Y_j. The space generated by A is the family (wu is a concatenation of words w and u)*

$$\{w_{n_1}(a_1) w_{n_2}(a_2) \dots w_{n_l}(a_l) : n_1 < n_2 < \dots < n_l, a_i \in \Sigma\}.$$

If your solution to Exercise 5 looks as I believe it should, it should easily give Theorem 6 as well. One can strengthen this theorem by considering a partition of $S(v)$ as well, and moreover allow more than one variable. There is also a joint generalization of Theorem 6 and Gowers' theorem.

Theorem 7 (van der Waerden)**.** *If $\mathbb{N} = \bigcup_{i=1}^{k} Y_i$, then there is i such that Y_i contains an arbitrarily long arithmetic progression.*

Exercise 8.

(1) *Find a partition of $\mathbb{N}$ into two pieces such that neither of the pieces contains an infinite arithmetic progression.*
(2) *Prove van der Waerden's theorem.*[2][3][4]

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[1] Hint: Use Exercise 1 and the machinery that we have developed in proving Hindman's and Gowers' theorems.

[2] Hint 1: Use Hales-Jewett theorem.

[3] Hint 2: To get an arithmetic progression of length 10, let $\Sigma = \{0, 1, \dots, 9\}$.

[4] Note: There is also a very nice direct proof using the machinery that we have developed. In a sense, proving van der Waerden's theorem using Hales–Jewett's theorem is shooting a fly with a cannon, but our main goal was Gowers' theorem and the machinery developed to prove it is more suitable for proving Hales–Jewett then for proving van der Waerden.