Intersection pairings on singular quotients

Lisa Jeffrey

(Joint work with Y.-H. Kiem, F. Kirwan, J. Woolf)

I. Introduction

If M is a symplectic manifold acted on by a compact Lie group K in a Hamiltonian way, the moment map

$$\mu : M \to \mathbf{k}^*$$

is K-equivariant and enables us to form symplectic quotients

$$M_\lambda = \mu^{-1}(\mathcal{O}_\lambda)/K$$

(where $\mathcal{O}_\lambda$ is the orbit of the coadjoint action through $\lambda \in \mathbf{k}^*$).

Special case: $M_0 = \mu^{-1}(0)/K$

If K abelian, our definition reduces to

$$M_\lambda = \mu^{-1}(\lambda)/K$$

Key fact: 0 regular value of μ iff M_0 is an orbifold (has at worst finite quotient singularities)

0 regular value iff K acts with finite stabilizers on $\mu^{-1}(0)$

Special case: K acts freely on $\mu^{-1}(0) \Longrightarrow M_0$ is a smooth manifold

In this talk we present some generalizations of results known in the case of orbifolds to more general symplectic quotients which may have more serious singularities.

II. Geometric invariant theory

M Kähler manifold (in particular symplectic) acted on by a complex group $G = K^{\mathbf{C}}$ preserving complex structure

Special case: $M \subset \mathbf{C}P^n$, $\mathbf{C}^{n+1}$ is a representation of G and hence G acts on $\mathbf{C}P^n$

Def. $\bullet x \in M$ *stable* iff $\mu^{-1}(0)$ meets Gx at a point whose stabilizer in K is finite

($M^s \stackrel{\text{def}}{=}$ stable points)

Def. $\bullet x \in M$ *semistable* iff $\mu^{-1}(0) \cap \overline{Gx} \neq \emptyset$

($M^{ss} \stackrel{\text{def}}{=}$ semistable points)

Def. The GIT quotient of M is $M//G = M^{ss}/G$

Prop. If all semistable points are stable, $M//G$ is an orbifold.

Thm. [Mumford, Guillemin-Sternberg, Kirwan, Kempf-Ness] $M//G = \mu^{-1}(0)/K$

(symplectic quotient = GIT quotient)

III. Kirwan map

Def. $H^*_K(M)$: K-equivariant cohomology of M

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If 0 is a regular value of μ, define the *Kirwan map*

$$\kappa : H^*_K(M) \to H^*_K(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/K)$$

Thm. [Kirwan] κ surjective.

If 0 is not a regular value of μ, there is a suitable G-equivariant blowup $\tilde{M}$ (the partial desingularisation of M: Kirwan) for which $\tilde{M}//G$ has only orbifold singularities (see next section.)

• If 0 singular value of μ, define Kirwan map as

$$\kappa : H^*_K(M^{ss}) \to H^*_K(\tilde{M}^{ss}) \cong IH^*(\tilde{M}//G)$$

$$\ldots \to IH^*(M//G)$$

Here, IH^* denotes intersection cohomology.

Def. [Kiem] A linear action of a group G on a complex vector space V is *weakly linearly balanced* if

$$dim_{\mathbf{C}}(V//G) < codim\ \phi^{-1}(v)$$

where $V//G$ is a complex cone with vertex v and ϕ is the quotient map $V \to V//G$.

Def. [Kiem] An action of a reductive group G on a complex manifold M is *weakly balanced* if for any subgroup R with fixed point set Z_R, the action of R on the fibres of the normal bundle to GZ_R^{ss} is weakly linearly balanced, where Z_R^{ss} is the intersection $M^{ss} \cap Z_R$ (and a similar condition is also imposed on certain subgroups of R and subspaces of Z^{ss} associated to them).

Example: Action of $\mathbf{C}^*$ on $\mathbf{CP}^n$ is weakly balanced iff numbers of positive and negative weights are equal

Example: The actions of $SU(2)$ and its maximal torus $U(1)$ on $\mathbf{C}P^n$ and $(\mathbf{C}P^1)^n$ are weakly balanced.

Example: More examples of weakly balanced actions are provided by moduli spaces of holomorphic vector bundles over a Riemann surface.

Thm. [Kiem] If the action of G on M is weakly balanced, we can split the Kirwan map κ to obtain a subspace V_M of $H^*_K(M^{ss})$ on which the restriction of κ is an isomorphism onto $IH^*(M//G)$.

The restriction of an element of $H^*_K(M^{ss})$ to $G \times_{N^R} Z_R^{ss}$

is an element of

$$H^*_{N_0^R/R}(Z_R^{ss}) \otimes H^*(BR),$$

where N_0^R is the identity component of the normalizer of R. Here, BR is the classifying space of R, so $H^*(BR)$ is the R-equivariant cohomology of a point.

Ex. When $R = \mathbf{C}^*$, $H^*(BR) = \mathbf{C}[\zeta]$ a polynomial ring with one generator ζ of degree 2.

The subspace V_M is defined by imposing an upper bound on the degree of the elements in $H^*(BR)$. We require that they only involve elements of degree $< n_R$ in $H^*(BR)$ (where n_R is the dimension of the reduced space of the normal bundle to GZ_R^{ss}).

Example: For the standard action of $K = SU(2)$ on $\mathbf{C}P^n$ (coming from the linear action on $\mathbf{C}^{n+1}$), the equivariant cohomology $H_T^*(M)$ of M with respect to the maximal torus T of $K = SU(2)$ is generated by two equivariant cohomology classes ξ and ζ, both of degree two, subject to the relation

$$(\xi - n\zeta)(\xi - (n-2)\zeta)...(\xi + (n-2)\zeta)(\xi + n\zeta) = 0.$$

$\zeta =$ generator of $H_T^*(\mathrm{pt})$

$\xi = [\omega + (\mu, \zeta)]$ is the extension of the symplectic form to a class in xequivariant cohomology.

The non-identity element of the Weyl group W of K sends ζ to $-\zeta$ and fixes ξ. Thus the equivariant cohomology $H_K^*(M)$ of M with respect to K is generated by ξ and ζ^2 subject to the same relation.

The subspace V_M is given by those elements for which the degree in ζ is $\leq \frac{n}{2} - 2$.

IV. Partial desingularization

Let $\mathcal{R}(r)$ be the set of conjugacy classes of subgroups of G of dimension r which are conjugate to the stabilizer of some $x \in M$. For each $R \in \mathcal{R}(r)$ we denote by Z_R^{ss} the fixed point set of R in M^{ss}.

Thm. [Kirwan] There exists a manifold $\tilde{M}$ which fibres over M in a G-equivariant way, for which $\tilde{M}//G$ has only orbifold singularities.

(It is possible to resolve the singularities of $\cup_{R \in \mathcal{R}(r)} G Z_R^{ss}$ equivariantly, to form a space $\hat{M}_1$ with an action of G covering the action on M. We perform this operation for the maximal value of r, and then all subgroups R which occur as stabilizers on $\hat{M}_1$ have dimension $< r$. We may repeat until we are left with a manifold $\tilde{M}$ with a G action for which only finite stabilizers occur.)

V. Reduction to abelian case

Thm. 1

$$\kappa_{\tilde{M}}(\eta)[\tilde{M}//G] = A\kappa^T_{M,\xi}(\eta\mathcal{D}^2)[M//_\xi T]$$

Note $M//_\xi T = \tilde{M}//_\xi T$ for generic values of ξ./
Here:

- A is a constant depending on G and the orders of the stabilizers in K and its maximal torus T at a generic point.
- $\tilde{M}^T$ is the partial desingularization of M for the action of T
- $\mathcal{D}$ is a fixed class in the T-equivariant cohomology of a point, which maps to $H^*_T(\tilde{M})$: it is the product $\Pi_{\gamma>0}\gamma$ of the positive roots. $\gamma \in (\mathrm{Lie}(T))^*) \subset H^*_T(\mathrm{pt})$
- ξ is a small constant in $\mathbf{t}$.

Since the quotient $M//_\xi T$ has only orbifold singularities, the right hand side can be computed in terms of the fixed point set of T on M using the residue formula (J-Kirwan '95).

VI. Pairings

$$H_K^*(M) \to H_K^*(\tilde{M}) \xrightarrow{\kappa} H^*(\tilde{M}//G)$$

Thm. 2 If α and β are in V_M, then the pairing of $\kappa(\alpha)$ and $\kappa(\beta)$ in $IH^*(M//G)$ is given by

$$\kappa_{\tilde{M}}(\alpha\beta)[\tilde{M}//G]$$
$$= A\kappa^T_{\tilde{M}^T}(\alpha\beta\mathcal{D}^2)[M//_\xi T]$$

This pairing is given by

$$A'\mathrm{Res}(\mathcal{D}^2 \sum_F \int_F \frac{\alpha\beta e^{\bar{\omega}-\epsilon}}{e_F}),$$

where A' is another constant (depending on G), and F runs over a subset of the components of the fixed point set of T acting on M, while ϵ is a small parameter in $\mathbf{t}$. The class e_F is the equivariant Euler class of the normal bundle to F in M. Also, Res is an iterated residue in l variables, where l is the rank of G.

Example

Consider the action of $K = SU(2)$ on $M = \mathbf{C}P^n$ (via the linear action on $\mathbf{C}^{n+1}$).

As noted above, the equivariant cohomology $H_T^*(M)$ of M with respect to the maximal torus T of $K = SU(2)$ is generated by two equivariant cohomology classes ξ and ζ, both of degree two, subject to the relation

$$(\xi - n\zeta)(\xi - (n-2)\zeta)...(\xi + (n-2)\zeta)(\xi + n\zeta) = 0,$$

where the nonidentity element of the Weyl group W of K sends ζ to $-\zeta$ and fixes ξ. Thus the equivariant cohomology $H_K^*(M)$ of M with respect to K is generated by ξ and ζ^2 subject to the same relation.

The fixed point sets for the action of T are the $n+1$ points represented by the weight vectors; the values taken by the moment map on the fixed points are just the weights $-n, 2-n, ..., n-2, n$ up to a universal scalar multiple.

If n is odd, 0 is a regular value of the moment map. If on the other hand n is even then if $\alpha, \beta \in H_K^*(M)$ satisfy $\deg(\alpha) + \deg(\beta) = 2(n-3)$ and if $\alpha|_{M^{ss}}$ and $\beta|_{M^{ss}}$ lie in the subspace V_M of $H_K^*(M^{ss})$ then

$$\langle \kappa_M(\alpha), \kappa_M(\beta) \rangle_{IH^*(M//G)} = \kappa_{\tilde{M}}(\alpha\beta)[\tilde{M}//G]$$

$$= \mathrm{Res}_{X=0} \sum_{j:0<n-2j\leq n} \frac{q((n-2j)X, X^2)}{(2X)^{n-2} \prod_{k\neq j}(k-j)}$$

where $\alpha\beta = q(\xi, \zeta^2)$.

VII. Witten's integral

Def.

$$I^\epsilon = \int_{X \in \mathbf{k}} dX e^{-\epsilon|X|^2/2} \int_M e^\omega e^{i\mu(X)}$$

If 0 is a regular value of μ,

$$I^\epsilon = I_0^\epsilon + O(e^{-b/\epsilon})$$

for some $b > 0$, where I_0^ϵ is a polynomial in ϵ.

This polynomial has a topological interpretation in terms of the intersection numbers of the reduced space M_0:

$$I_0^\epsilon = \int_{M_0} e^{\epsilon\Theta} e^\omega$$

where $\Theta \in H^4(M_0)$ is a characteristic class of the principal bundle $\mu^{-1}(0) \to M_0$.

It can be computed using a formula (the residue formula) involving fixed point data for the T action on M. (J-Kirwan '95)

On the other hand:

If 0 is a critical value of μ,

$$I^\epsilon = I_0^\epsilon + O(e^{-b/\epsilon})$$

where I_0^ϵ is a polynomial in $\sqrt{\epsilon}$. There is a residue formula for I_0^ϵ, which now may involve components F of the fixed point set for which $\mu(F) = 0$.

The interpretation of I_0^ϵ in terms of topological information about singular symplectic quotients is as yet unknown.