Symplectic Topology, Geometry and Gauge Theory

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Symplectic geometry has its roots in classical mechanics.

A prototype for a symplectic manifold is the *phase space* which parametrizes the position q and momentum p of a classical particle.

If the Hamiltonian (kinetic + potential energy) is

$$H = \frac{p^2}{2} + V(q)$$

then the motion of the particle is described by *Hamilton's equations*

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q}$$

In mathematical terms, a symplectic manifold is a manifold M endowed with a 2-form ω which is:

• closed: $d\omega = 0$

(integral of ω over a 2-dimensional submanifold which is the boundary of a 3-manifold is 0)

• nondegenerate (at any $x \in M$ ω gives a map from the tangent space $T_x M$ to its dual $T_x^* M$; nondegeneracy means this map is invertible)

If $M = \mathbf{R}^2$ is the phase space equipped with the Hamiltonian H then the symplectic form $\omega = dq \wedge dp$ transforms the 1-form

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq$$

to the vector field

$$X_H = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$$

The flow associated to X_H is the flow satisfying Hamilton's equations.

Darboux's theorem says that near any point of M there are local coordinates

$$x_1, \ldots, x_n, y_1, \ldots, y_n$$

such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

(Note that symplectic structures only exist on even-dimensional manifolds)

Thus there are no local invariants that distinguish between symplectic structures: at a local level all symplectic forms are identical.

In contrast, Riemannian metrics g have local invariants, the curvature tensors $R(g)$, such that if $R(g_1) \neq R(g_2)$ then there is no smooth map that pulls back g_1 to g_2.

Geometries: For $x \in M$, we specify

• Riemannian geometry: Riemannian metric g, an inner product on $T_x M$

• Algebraic geometry: (Almost) complex structure J, a linear map $J : T_x M \to T_x M$ satisfying $J^2 = -1$

• Symplectic geometry: a skew symmetric bilinear form ω on $T_x M$ which is invertible as a map $T_x M \to T_x^* M$

We may require that any two of these structures be compatible:

Ex. ω is compatible with J if $J^* \circ \omega \circ J = \omega$

Any two of these which are compatible determine the third. The intersection is (almost) *Kähler* geometry: an (almost) Kähler manifold is an (almost) complex manifold equipped with a compatible symplectic form and metric.

Complex and almost complex structures:

An almost complex structure J is *integrable* (comes from a structure of complex manifold) if the *Nijenhuis tensor* N vanishes. One way to express this condition is that the tangent space of a manifold with an almost complex structure J may be decomposed into $\pm\sqrt{-1}$ eigenspaces, and hence any vector field X decomposes as the sum of a $+\sqrt{-1}$ part X' and a $-\sqrt{-1}$ part X''. The condition that J is integrable means that if X and Y are two vector fields then the Lie bracket $[X', Y']$ of the $+\sqrt{-1}$ parts is again a vector field in the $+\sqrt{-1}$ eigenspace of $TM \otimes \mathbf{C}$.

A complex manifold is a manifold equipped with coordinate charts $z_1, \ldots, z_n$, where $z_j \in \mathbf{C}$, for which the chart maps are holomorphic maps (in all the variables z_i). A complex manifold automatically has an almost complex structure (if $z_j = x_j + \sqrt{-1}y_j$, we require that J identifies with the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ if the tangent space is identified with (x, y).)

An almost complex structure is integrable iff it comes from a structure of complex manifold.

For any symplectic manifold there exists an almost complex structure compatible with the symplectic structure. This need not be a complex structure (many symplectic manifolds cannot be given a structure of complex manifold).

Integrable systems (Workshop 1):

An integrable system is a collection of n smooth functions $f_1, \ldots, f_n$ on a symplectic manifold of dimension $2n$ for which the Hamiltonian vector fields X_j are linearly independent almost everywhere and satisfy

$$\omega(X_i, X_j) = 0 \quad \forall i, j.$$

Some notable examples were treated in this workshop (Calogero-Moser system, Toda lattice,...)

Symplectic and contact topology (Workshop 2) In symplectic topology one of the major concerns is the study of maps from one symplectic manifold to another to obtain topological information. The focus may take several forms.

1. Symplectic capacities: Roughly, symplectic capacities measure the size of a ball in $\mathbf{R}^{2n}$ which may be embedded symplectically in a symplectic manifold.

In dimension 2, the symplectic form is simply the area or volume form, so a map preserving the symplectic form is simply a volume-preserving map. In higher dimensions, however, the symplectic form imposes a stronger constraint than that a map preserve the volume.

Example: Symplectic camel problem [McDuff, Traynor]

Let $M = \mathbf{R}^{2n} - H$, where H is a hyperplane with a small ball U removed (so M is connected).

Question: is there a continuous family $(F_t : t \in [0,1])$ of symplectic embeddings of the unit ball B into M for which $F_0(B)$ is entirely on one side of H and $F_1(B)$ is on the other side of H?

Answer: Not unless the radius of the ball U is large enough. (In contrast it is easy to find a volume-preserving map which will do this.)

Another way to say this is that the symplectic capacity of M is finite. (Of course the symplectic capacity of $\mathbf{R}^{2n}$ is always infinite.)

2. Pseudoholomorphic curves:

We study the space $\mathcal{M}$ of smooth maps

$$f : S^2 \to M$$

for which

$$f_* \circ J_{S^2} = J_M \circ f_*$$

where $f_* : T_x S^2 \to T_{fx} M$.

This can be used to define invariants generalizing topological invariants from (co)homology.

Intersection numbers:

Y_j submanifolds of M^m of dimension r_j with $\Sigma_j (m - r_j) = m$ so the dimension of $\cap_j Y_j$ is 0 (if in general position) The intersection number of the Y_j is the number of points in $\cap_j Y_j$ (counted with appropriate signs associated to the orientations of the Y_i).

By Poincaré duality any class Y in M determines a cohomology class α_Y on M. If the dimensions of the α_Y add up to the dimension of M then we can define

$$\int_M \cup_j \alpha_{Y_j}.$$

To the α_Y we may associate topological invariants of the space $\mathcal{M}$ (the *Gromov-Witten invariants*). These generalize the intersection numbers (the latter correspond to the component of the moduli space $\mathcal{M}$ corresponding to those maps which send S^2 to a point).

Analogues and generalizations of the Gromov-Witten invariants also formed the subject of the Coxeter lecture series by A. Givental; Givental spoke on his proof of the Virasoro conjecture, namely that certain generating functions which encode the Gromov-Witten invariants are annihilated by the generators of the Virasoro algebra (the algebra of generators of the Lie algebra of the loop group).

3. Arnol'd conjecture: The Arnol'd conjecture states that any diffeomorphism ϕ on a symplectic manifold M arising from the flow of a (time dependent) Hamiltonian function H_t has at least as many fixed points as a smooth function on M must have critical points (in other words the number of fixed points is at least $\Sigma_j b_j$ where b_j is the dimension of the j'th cohomology group).

The Arnol'd conjecture has been proved under the hypothesis that $\pi_2(M) = 0$ or more generally $\omega[\pi_2(M)] = 0$. Recently this hypothesis has been eliminated (Fukaya-Ono).

4. Gauge theory

A connection A on a vector bundle E of rank n over a manifold M may be thought of as an element of

$$\mathcal{A} = \Omega^1(M) \otimes \mathrm{End}(\mathbf{C}^n)$$

(once we have trivialized E). The curvature is

$$F_A = dA + A \wedge A \in \Omega^2(M) \otimes \mathrm{End}(\mathbf{C}^n)$$

We may study the critical points of a functional Φ on $\mathcal{A}$, for example the *Yang-Mills functional*

$$\Phi(A) = \int_M |F_A|^2.$$

Each critical manifold is infinite-dimensional.

Gauge group: $\mathcal{G}$ is the space of automorphisms of a principal G-bundle $P \to M$ covering the identity map on M.

If we have chosen a trivialization of P,

$$\mathcal{G} = C^\infty(M, G)$$

(choice of trivialization = “choice of gauge”)

Infinite-dimensional Lie groups and gauge theory formed the subject of a graduate course by B. Khesin which was taught as part of the thematic program.

Morse theory:

M Riemannian manifold with a smooth function

$$F : M \to \mathbf{R}.$$

Critical manifolds are the submanifolds where the gradient ∇F of F vanishes. (e.g. maximum, minimum)

Gradient flow

$$\frac{d\Phi}{dt} = \nabla F(\Phi(t))$$

The gradient flow paths define a differential on a chain complex for which the chains are linear combinations of critical manifolds.

This picture generalizes to some functionals on infinite dimensional manifolds.

(a) Symplectic action functional:

(Symplectic Floer homology)

$(Y.\omega)$ symplectic manifold

$M = C^\infty(S^1, Y)$ loops in Y

Define symplectic action functional on M by

$$F(\gamma) = \int_{\tilde{\gamma}} \omega$$

where $\tilde{\gamma} : 2 - \text{disc} \to Y$ is a disc whose boundary is the image of γ.

If ω is integral, F gives a well defined map from M to $\mathbf{R}/\mathbf{Z}$

If $\int_Z \omega = 0$ for any closed submanifold $Z \cong S^2$, the map is in fact well defined into $\mathbf{R}$.

Symplectic Floer homology is the Morse theory of this function F on the loop space of Y.

It is an analogue of *instanton Floer homology*, which is the Morse theory of the Yang-Mills functional.

5. Contact manifolds

α 1-form on (2n+1)-dimensional manifold Y for which

$$\alpha \wedge (d\alpha)^n \neq 0$$

A *contact structure* is the distribution on Y given by the kernel of α. (Different forms α give rise to the same distribution)

Contact structures occur on odd-dimensional manifolds. It is natural to study manifolds with boundary for which the interior of the manifold has a symplectic structure and the components of the boundary have contact structures.

This gives rise to *symplectic field theory* (as in Y. Eliashberg's minicourse).

Hamiltonian group actions (Workshop 3)

Let M be a symplectic manifold equipped with a group action preserving the symplectic structure

e.g. $M = \mathbf{C}^n$, $U(1)$ acts by diagonal multiplication

$$e^{i\theta} \in U(1) : (z_1, \ldots, z_n) \mapsto (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$$

Under this action, every element of the Lie algebra gives rise to a vector field on M.

The action is *Hamiltonian* if these vector fields are Hamiltonian vector fields for some functions on M which fit together to give a map $\mu : M \to \mathrm{Lie}(G)$

(If we introduce a basis for Lie(G), the j'th component of μ is the Hamiltonian function giving rise to the vector field corresponding to the j-th component of Lie(G).)

This map is called the *moment map*.

In addition we require that the moment map be equivariant with respect to the adjoint action on Lie(G) (if G is abelian this simply means the map is invariant under the group action).

The moment map can be used to divide out the group action to obtain another symplectic manifold, the symplectic quotient. (If the group is odd-dimensional e.g. U(1), the topological quotient by the group action would also be odd-dimensional and hence could not be symplectic.)

The symplectic quotient is defined as

$$M_{\mathrm{red}} = \mu^{-1}(0)/G;$$

this is a smooth manifold if G acts freely on $\mu^{-1}(0)$.

For example, for the diagonal action of $U(1)$ on $\mathbf{C}^n$ the symplectic quotient is complex projective space $\mathbf{C}P^{n-1}$.

Hamiltonian group actions formed the subject of a graduate course by L. Jeffrey which was taught as part of the thematic program.

1.Equivariant cohomology of symplectic manifolds: If a symplectic manifold has a Hamiltonian group action it is natural to study its equivariant cohomology

$$H^*(M \times_G EG).$$

The symplectic form extends naturally to give a class in equivariant cohomology.

2. Flat connections on Riemann surfaces:

The moduli space of flat connections on a Riemann surface up to equivalence under the action of the gauge group (or more precisely the space of flat connections on a punctured Riemann surface with central holonomy around the puncture) is a symplectic manifold.

The dimensions of its cohomology groups were determined by Atiyah and Bott in a seminal 1982 paper.

At this workshop, S. Tolman and J. Weitsman gave a proof of the formula for the dimensions of the cohomology groups of this space, using Morse theory by constructing a perfect Morse function (one for which the Morse inequalities are equalities).

S. Racaniere spoke on the generators of the cohomology ring of these moduli spaces.

Y.-H. Kiem spoke on the cohomology of these spaces, relaxing the hypothesis that they must be smooth manifolds.

A. Szenes spoke on the quantization of these moduli spaces (i.e. the Verlinde formula).

Flat connections on Riemann surfaces formed the subject of a graduate course by E. Meinrenken which was taught as part of the thematic program.

3. Toric geometry

A particularly nice class of symplectic manifolds equipped with Hamiltonian group actions are the toric manifolds. These are manifolds of dimension $2n$ equipped with a Hamiltonian action of $U(1)^n$. The image of the moment map is a polytope (the *Newton polytope*). The components of the moment map form an integrable system; the symplectic quotients of a toric manifold is a point.

Several talks treated toric manifolds; M. Abreu spoke on Kähler metrics on toric manifolds, and Y. Karshon spoke on geometric quantization of toric manifolds.

4. Geometric quantization

In quantum mechanics, we replace the phase space M (parametrizing position and momentum) by the space of wave functions, the square integrable functions on the configuration space (the space parametrizing positions of a particle).

The mathematical generalization of this procedure is to associate to a symplectic manifold M the space of sections of a complex line bundle over M ("prequantum line bundle with connection") which in some way depend on only half the variables parametrized by M. A natural way to implement this is to require that the sections be holomorphic sections of the line bundle.

Quantization was treated in several talks (Y. Karshon, G. Landweber, S. Wu, P. Xu).

5. Morse theory and graphs

Several talks (V. Guillemin, C. Zara; T. Holm) treated the cohomology of a class of spaces with group actions known as

Goresky-Kottwitz-MacPherson spaces.

These are spaces equipped with an action of $T = U(1)^n$ for which the one-skeleton is of dimension at most 2. (The one-skeleton is defined to be the space of points x for which the T-orbit through x has dimension at most 1.)

6. Group-valued moment map spaces

Several talks (S. Racaniere, A. Alekseev) treated the group-valued moment map spaces of Alekseev, Malkin and Meinrenken. These spaces are equivalent to spaces with the Hamiltonian action of a loop group; they are equipped with the action of a Lie group G, but the analogue of the moment map takes values in G itself rather than in its Lie algebra.

7. Poisson-Lie groups and representation theory

Several talks (P. Xu, M. Kogan, S. Evens, A. Alekseev) treated Poisson-Lie groups. In many cases (e.g. B. Kostant. A. Knutson, R. Brylinski, P. Paradan) the talks proved results in representation theory of Lie groups.

8. Yang-Baxter equation, R-matrices, integrable systems

Several talks (A. Alekseev, J. Hurtubise, E. Markman, R. Donagi) treated R-matrices and integrable systems associated to them.