Exact Simulation-Based Tests in Multivariate Regressions: Applications to Asset Pricing Models [1]

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December 2000, compiled December 24, 2000

[1]This work was supported by the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds FCAR (Government of Québec).

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Abstract

Multivariate regressions (MR) are among the simplest empirical models of financial econometrics. It is well known however that despite their simple statistical structure, standard asymptotically justified MR-based tests are unreliable. Exact tests have been proposed for a few specific hypotheses [e.g. Gibbons, Ross and Shanken (Econometrica 1989), Shanken (Journal of Finance 1986), Velu and Zhou (Journal of Empirical Finance 1999), Stewart (Econometric Reviews 1997)], most of which depend on normality. In this paper, we propose likelihood based exact market-model tests for possibly non-linear hypotheses, allowing for a wide class of error distributions which include normality as a special case. The proposed test procedures are computationally attractive and may be easily obtained by simulation. For the Gaussian model, our test results serve to unify existing results on efficiency tests. In non-Gaussian contexts, we re-consider efficiency tests allowing for multivariate student-t errors and unknown zero-beta rate. In this case, we propose a set estimate for the intervening degrees-of-freedom parameter, which serves to devise a confidence-set based exact Monte Carlo test.

1 Introduction

Multivariate regressions (MR) are among the simplest empirical models of financial econometrics. The most familiar MR applications in finance include *market-models* such as the capital asset pricing model (CAPM), which may be traced back to Gibbons (1982) and Gibbons, Ross and Shanken (1989, GRS) [GRS]. The associated empirical literature which has evolved since Gibbons' seminal work is enormous; recent references can be found in Campbell, Lo and MacKinlay (1997).

Despite the simplicity of their statistical structure, there are good reasons for skepticism regarding the reliability of standard test procedures in MR contexts. Indeed, a number of Monte Carlo studies (see for example Dufour and Khalaf (1998), Stewart (1997), Campbell et al. (1997) and the references cited therein) have provided extensive evidence that MR-tests can be severely biased towards over-rejection. These difficulties find their origin in the fact that the intervening exact null distributions typically depend on nuisance parameters - *e.g.* the error covariance parameters - whose number increases rapidly with the system's dimension.

Although most MR-tests are large-sample based, exact tests have been proposed for a few specific hypotheses. Well known procedures include the exact efficiency test proposed by Gibbons et al. (1989) for the observable risk-free rate case. Specifically, GRS used the Hotelling-T^2 statistic, which may be transformed into an F-distributed statistic, to test the joint significance of the MR-CAPM intercepts. See also Stewart (1997) for more recent work on exact F tests in finance.

In CAPM contexts which assume unobservable risk-free rate, two exact bounds tests have been independently proposed for the same (non-linear) problem, namely testing the efficiency hypothesis in a multi-factor MR-CAPM. The first bound is due to Shanken (1986) and has recently been the subject of renewed attention; see Campbell et al. (1997) (chapter 5). The second bound which is developed in Stewart (1997) is directly based on Dufour (1989)'s conservative bound test. The relation between these bounds remains unexplored. Zhou (1991) and **?** proposed a bounds test which is tighter than the latter two, yet depends importantly on Gaussianity.

As a matter of fact, all the exact distributional results just cited are conditional on the normality assumption. Zhou (1993) reconsidered the GRS problem under elliptical distributions and provided simulation-based test procedures which exploit exact invariance results. Although nuisance parameters are not completely accounted for by Zhou (1993), to the best of our knowledge, no other exact results are available which do not impose gaussianity.

In a different vein, Dufour and Khalaf (1998) have recently proposed several general exact test procedures for MR models. In particular, they considered the Wilks statistic, defined as the ratio of the determinants of the constrained and unconstrained sum of squared error matrices. For a specific class of hypotheses which take the Uniform Linear (UL) form (see e.g. Stewart (1997)), they propose an exact simulation-based Wilks test without the normality assumption. They also show how to obtain an exact bounds Wilks test for general, possibly non-linear non-gaussian hypotheses. In this paper, we discuss how to apply these results to asset pricing tests. We restrict primary focus to the efficiency hypothesis, in view of its fundamental importance.

The paper makes three main contributions. First, we show that both Shanken's and Stewart's bounds obtain as a special case of the bound proposed by Dufour and Khalaf (1998). Our analysis further reveals that Shanken's and Stewart's bounds are in fact equivalent, a point which seems to have escaped notice. Secondly, we extend the latter efficiency tests beyond the Gaussian model; the elliptical distributions assumed by Zhou (1993) are included as a special case. Thirdly, for the multivariate student t-error distribution, we propose a formal method which deals with the degrees-of-freedom nuisance parameter problem. To do this, we provide an exact set estimate for the parameter. The latter is then used to obtain a confidence-set-based exact Monte Carlo test; see Dufour and Kiviet (1996) and Dufour (1995).

The tests proposed are applied to market efficiency problems with unknown risk-free rates. We consider monthly returns on New York Stock Exchange (NYSE) portfolios, which we construct from the University of Chicago Center for Research in Security Prices (CRSP) 1926-1995 data base.

The paper is organized as follows. Section 2 briefly reviews the exact MR tests proposed by Dufour and Khalaf (1998). Section 3 considers the existing efficiency tests and suggests extensions to non-normal models. Section 4 presents the case of multivariate t-errors with unknown degrees of freedom. In Section 5, we report the empirical application and section 6 concludes.

2 General multivariate regressions tests

The multivariate regression (MLR) model is of the form

$$Y = XB + U \tag{2.1}$$

where $Y = [Y_1, \ldots, Y_p]$ is $n \times p$, X is $n \times K$ with rank K and is assumed fixed, and $U = [U_1, \ldots, U_p] = [\widetilde{U}_1, \ldots, \widetilde{U}_n]'$ is an $n \times p$ matrix of random disturbances. For further reference, let $B = [b_1, \ldots, b_p]$, $b_j = (b_{0j}, b_{1j} \ldots, b_{sj})'$, $j = 1, \ldots, p$ where $s = K - 1$. The bounds derived by Dufour and Khalaf (1998) further assume

$$\widetilde{U}_i = JW_i \ , \ i = 1, \ldots, n \, , \tag{2.2}$$

where the vector $w = vec(W_1, \ldots, W_n)$ has a known distribution and J is an unknown, non-singular matrix. In this context, the covariance matrix of $\widetilde{U}_i$, which is denoted Σ, is $JJ'$and is invertible. The latter distributional assumption includes normality as a special case. It is well known that in this context, the OLS estimator

$$\widehat{B} = (X'X)^{-1}X'Y$$

corresponds to the Gaussian unconstrained maximum likelihood estimator (MLE). For further reference, let

$$\widehat{\Sigma} = (Y - X\widehat{B})'(Y - X\widehat{B})/n$$

denote the associated (unconstrained) estimate of Σ.

For convenience, rewrite the model as

$$y = (I_p \otimes X)b + u \tag{2.3}$$

where $y = vec(Y)$, $b = vec(B)$, and $u = vec(U)$, and consider the general hypothesis

$$H_0 : \mathfrak{R}b \in \Delta_0 \tag{2.4}$$

where $\mathfrak{R}$ is a $q \times (pK)$ matrix of rank q, and Δ_0 is a non-empty subset of $\mathbb{R}^q$. The associated LR statistic is:

$$LR = n\ln(\mathfrak{L}) \ , \quad \mathfrak{L} = |\widehat{\Sigma}_0|/|\widehat{\Sigma}| \tag{2.5}$$

where $\widehat{\Sigma}_0$ is the constrained MLE of Σ. The statistic $\mathfrak{L}$ corresponds to the inverse of the well known Wilks statistic.

In the present paper, we will exploit the following distributional results from Dufour and Khalaf (1998) pertaining to (2.4) and a special case of the latter which takes the following form

$$H_{01} : \mathcal{R}B\mathcal{C} = \mathcal{D} \tag{2.6}$$

where $\mathcal{R}$ is an $r\times K$ matrix of rank r, $\mathcal{C}$ is a $p\times c$ matrix of rank c. On observing that (2.6) corresponds to $(\mathcal{C}' \otimes \mathcal{R})\, b = vec(\mathcal{D})$, it is clear that not all linear hypotheses can be cast in the UL form.

Theorem 1 *Under* (2.1), (2.2) *and* (2.6), *Wilks' statistic*

$$\Lambda = |\widehat{\Sigma}_{01}|/|\widehat{\Sigma}| \tag{2.7}$$

where $\widehat{\Sigma}_{01}$ *and* $\widehat{\Sigma}$ *are constrained unconstrained MLE of* Σ, *is distributed like*

$$\left|W'\widetilde{M}W\right| / \left|W'\widetilde{M}_0W\right| \tag{2.8}$$

with

$$\begin{aligned}
\widetilde{M}_0 &= \widetilde{M} - \widetilde{X}(\widetilde{X}'\widetilde{X})^{-1}\mathcal{R}'[\mathcal{R}(\widetilde{X}'\widetilde{X})^{-1}\mathcal{R}']^{-1}\mathcal{R}(\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}', \\
\widetilde{M} &= I - \widetilde{X}(\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}, \\
\widetilde{X} &= X\mathcal{C},
\end{aligned}$$

and $W = [W_1, \ \ldots \ , \ W_p]$ *is defined by* (2.2).

Now using the latter result, it is easy to obtain simulated values of the test statistic under the null hypothesis and (2.2). These may be used to obtain an exact test, as follows (see also Dufour and Khalaf (1999)).

1. Let Λ_0 denote the observed test statistic.

2. By Monte Carlo methods and for a given number N of replications, draw $W^j = [W_1^j, \ldots, W_p^j]$, $j = 1, ..., N$, conforming with (2.2).

3. From each simulated error matrix W^j, compute the statistics
$$\left|W^{j\prime}\widetilde{M}W^j\right| / \left|W^{j\prime}\widetilde{M}_0W^j\right|, \; j = 1, \ldots, N,$$
as defined in Theorem 1, (2.8).

4. Compute the rank $\widehat{R}_N(\Lambda_0)$ of Λ_0 in the series $\Lambda_0, \; \Lambda_1, \; ..., \; \Lambda_N$. Then reject the null hypothesis at level α, when
$$\begin{aligned} \widehat{p}_N(\Lambda_0) &\leq \alpha, \\ \widehat{p}_N(\Lambda_0) &= 1 - \frac{\widehat{R}_N(\Lambda_0) - 1}{N+1} . \end{aligned} \tag{2.9}$$

For certain values of r and c and normal errors, the null distribution in question reduces to the F distribution. For instance, if $\min(r, c) \leq 2$, then

$$\left[(\Lambda^{1/\tau} - 1)\right] \frac{\rho\tau - 2\lambda}{rc} \sim F(rc\,, \rho\tau - 2\lambda) \tag{2.10}$$

where

$$\begin{aligned} \lambda &= \frac{rc-2}{4} \\ \rho &= n - K - \frac{(c-r+1)}{2}, \\ \tau &= \begin{cases} \left((r^2c^2 - 4)/(r^2 + c^2 - 5)\right)^{1/2} & , \; if \;\; r^2 + c^2 - 5 > 0 \\ 1 & , \; otherwise \end{cases} . \end{aligned}$$

Further, the special case $r = 1$ leads to the Hotelling's T^2 criterion which is a monotonic function of Λ. If $r > 2$ and $c > 2$, then the distributional result (2.10) holds asymptotically [Rao (1973, Chapter 8)]. Stewart (1997) provides an extensive discussion of these special F tests.

Of course, these results are restricted to UL hypotheses of the form (2.2). It is well known however that beside this specific hypothesis class, the null distribution of the LR statistic is not nuisance-parameter-free. Let us now consider the general hypothesis H_0 as defined by (2.4).

Theorem 2 *Under* (2.1), (2.2) *and* (2.4), *the null distribution of Wilks' statistic* $\mathfrak{L}$ *defined by* (2.5) *may be bound as follows*

$$P[\mathfrak{L} \geq x] \leq P[\Lambda_* \geq x], \ \forall x,$$

$$\Lambda_* = |\widehat{\Sigma}_{02}|/|\widehat{\Sigma}| \ , \tag{2.11}$$

where $\widehat{\Sigma}$ *is the unconstrained MLE,* $\widehat{\Sigma}_{02}$ *is the MLE under UL restrictions of the form*

$$H_{02} : \mathcal{R}_* B \mathcal{C}_* = \mathcal{D}_* \tag{2.12}$$

$\mathcal{R}_*$ *is an* $r_* \times K$ *matrix of rank* r_*, $\mathcal{C}_*$ *is a* $p \times c_*$ *matrix of rank* c_*, *and*

$$H_{02} \subseteq H_{01}. \tag{2.13}$$

In other words, if $\lambda_*(\alpha)$ is the α-level cut-off point obtained (as in Theorem 1) such that $P[\Lambda_* \geq \lambda_*(\alpha)] = \alpha$, then under the null hypothesis $P[\mathfrak{L} \geq \lambda_*(\alpha)] \leq \alpha$. For further reference, we will call H_{02} the "bounding null". Λ_* is the inverse of the Wilks' statsicti to test H_{02}, which is UL (i.e. of the form (2.6)). It is easy to see that (2.13) implies

$$\mathfrak{L} \leq \Lambda_* \tag{2.14}$$

which establishes the desired bound. For simulation purposes, it is useful to rewirte the latter resut using Theorem 1, as follows.

Corollary 3 *Under* (2.1), (2.2) *and* (2.4), *the null distribution of Wilks' statistic* $\mathfrak{L}$ *defined by* (2.5) *may be bound as follows*

$$P[\mathfrak{L} \geq x] \leq P\left[\frac{|WM^*W|}{|W'M_0^*W|} \geq x\right], \ \forall x, \tag{2.15}$$

where

$$\begin{aligned} M_0^* &= M - \widetilde{X}_*(\widetilde{X}_*'\widetilde{X}_*)^{-1}\mathcal{R}_*'[\mathcal{R}_*(\widetilde{X}_*'\widetilde{X}_*)^{-1}\mathcal{R}_*']^{-1}\mathcal{R}_*(\widetilde{X}_*'\widetilde{X}_*)^{-1}\widetilde{X}_*', \\ M^* &= I - \widetilde{X}_*(\widetilde{X}_*'\widetilde{X}_*)^{-1}\widetilde{X}_*', \\ \widetilde{X}_* &= X\mathcal{C}_*, \end{aligned}$$

$\mathcal{R}_*$ *is an* $r_* \times K$ *matrix of rank* r_*, $\mathcal{C}_*$ *is a* $p \times c_*$ *matrix of rank* c_*, *which satisfy*

$$\begin{aligned} H_{02} \ &: \ \mathcal{R}_* B \mathcal{C}_* = \mathcal{D}_*, \\ &H_{02} \subseteq H_{01}, \end{aligned}$$

and $W = [W_1, \ \ldots \ , \ W_p]$ *is defined by* (2.2).

A bounds MC p-value (BMC) may then be obtained by simulation under (2.2), as follows.

1. Let $\mathfrak{L}_0$ denote the observed test statistic.

2. By Monte Carlo methods and for a given number N of replications, draw $W^j = [W_1^j, \ \dots \ , \ W_p^j]$, $j = 1, ..., N$, conforming with (2.2).

3. From each simulated error matrix W^j, compute the "bounding" statistics $\left|W^{j\prime} M^* W^j\right| / \left|W^{j\prime} M_0^* W^j\right|$, $j = 1 \ , \ \dots \ , \ N$, as defined in Corollary 3, (2.15).

4. Obtain the rank $\widehat{R}_N^*(\mathfrak{L}_0)$ of $\mathfrak{L}_0$ in the series $\mathfrak{L}_0, \ \Lambda_{*1}, \ ..., \ \Lambda_{*N}$. Then the bounds α-level critical region corresponds to

$$\begin{aligned} \widehat{p}_N^*(\mathfrak{L}_0) &\leq \alpha, \\ \widehat{p}_N^*(\mathfrak{L}_0) &= 1 - \frac{\widehat{R}_N^*(\mathfrak{L}_0) - 1}{N+1} \ . \end{aligned} \tag{2.16}$$

In Gaussian models where $\min(r, c) \leq 2$, the bounds p-value may be obtained from the F distribution using (2.10).We proceed next to present the bounds tests proposed by Shanken (1986) and Stewart (1997) in CAPM contexts and show that these bounds obtain straightforwardly using the BMC test strategy underlying Theorem 2.

3 Efficiency tests: the case where error distributions are specified up to a scale matrix

A fundamental problem in financial economics involves testing the efficiency of a candidate benchmark portfolio. Let R_{ij}, $j = 1, \ \dots \ , \ p$, be returns on p securities for period i, $i = 1, \ ... \ , n$ and $\widetilde{R}_{ik}$, $k = 1, \ ... \ , s$ the returns on the s benchmark portfolios under consideration. A crucial assumption which further determines the econometric problem is whether the riskless rate of returns are observable or need to be estimated form the data. We consider both models in what follows. Throughout this section, we impose (2.2).

3.1 Observable risk-free rate

If it is assumed that a riskless asset R^F exists, then efficiency can be tested using the MLR model (2.1) with $Y = [r_1, \ ... \ , \ r_p]$, $X = [\iota_n, \ \widetilde{r}_1, \ ... \ , \ \widetilde{r}_s]$, where $r_j = (r_{1j}, \ ... \ , \ r_{nj})'$, $\widetilde{r}_k = (\widetilde{r}_{1k}, \ ... \ , \ \widetilde{r}_{nk})'$ and $r_{ij} = R_{ij} - R_i^F$, $\widetilde{r}_{ik} = \widetilde{R}_{ik} - R_i^F$, or alternatively:

$$r_{ij} = b_{0j} + \textstyle\sum_{k=1}^{s} b_{kj} \widetilde{r}_{ik} + U_{ij}, \qquad i = 1, ..., n, \quad j = 1, ..., p \ . \tag{3.1}$$

The hypothesis of efficiency implies that

$$H_{LCAPM} : b_{0j} = 0, \ j = 1, \ \dots \ , \ p, \tag{3.2}$$

i.e. the model intercepts b_{0j} are jointly equal to zero, which may be expressed, in UL form, as follows:

$$(1, 0, ..., 0)B = 0. \tag{3.3}$$

With respect to the above general framework, observe that H_{LCAPM} (3.3) is of the UL form (2.6) with $\mathcal{C} = I_p$, $\mathcal{D} = 0$ and $\mathcal{R}$ is the K-dimensional row vector $(1, 0, ..., 0)$. Conformably, the LR statistic to test (3.3) obtains as in (2.7)

$$LR = n \ln(\Lambda), \quad \Lambda = |\widehat{\Sigma}_{LCAPM}|/|\widehat{\Sigma}| \tag{3.4}$$

where $\widehat{\Sigma}_{LCAPM}$ is the MLE under H_{LCAPM} (3.3). Then using Theorem 1, the null distribution of LR can be characterized for all error distributions which satisfy (2.2), as follows.

Theorem 4 *Under* (3.1), (2.2) *and* (3.2), *the LR statistic defined by* (3.4) *is distributed like*

$$n \ln(\left|W'MW\right| / \left|W'M_0W\right|)$$

where $W = [W_1, \ \ldots \ , \ W_p]$ *is defined by* (2.2)

$$\begin{aligned} M &= I - X(X'X)^{-1}X', \\ M_0 &= M - X(X'X)^{-1}\mathcal{R}'[\mathcal{R}(X'X)^{-1}\mathcal{R}']^{-1}\mathcal{R}(X'X)^{-1}X', \end{aligned}$$

and $\mathcal{R}$ *is the K-dimensional row vector* $(1, 0, ..., 0)$.

Alternatively, of course, M_0 may be obtained as $I - X_s(X_s'X_s)^{-1}X_s'$, where $X_s = [\widetilde{r}_1, \ ... \ , \ \widetilde{r}_s]$. Based on the latter Theorem, as shown above, a MC p-value may be obtained by simulation for error distributions which satisfy (2.2). Recall that to obtain the MC p-value, one only needs to simulate the matrix $W = [W_1, \ \ldots \ , \ W_n]'$ which underlies (2.2). Here, an important special case is the multivariate t-distribution with known degrees-of-freedom. To draw W from a multivariate t with κ degrees of freedom, one may proceed as follows. Generate W_i (the rows of W) independently as

$$W_i = \mathcal{Z}_i/(\mathcal{C}_i/\kappa)^{1/2} \tag{3.5}$$

where $\mathcal{Z}_i$ is multivariate normal $(0, I_p)$ and $\mathcal{C}_i$ is a $\chi^2(\kappa)$ variate independent from $\mathcal{Z}_i$.

Under Gaussian errors, Theorem 4 and (2.10) imply that

$$(\Lambda - 1)\frac{(n-s-p)}{p} \sim F(p, n-s-p)\,,$$

which yields the Hotelling T^2 test proposed by Gibbons et al. (1989). Specifically, GRS suggest the following test statistic:

$$Q = \frac{n\widehat{\alpha}'\widehat{S}^{-1}\widehat{\alpha}}{1 + \overline{r}'\widehat{\Delta}^{-1}\overline{r}} \tag{3.6}$$

where $\widehat{\alpha}$ is the vector of intercept OLS estimates, $\widehat{S} = \frac{n}{n-K}\widehat{\Sigma}$ is the OLS-based unbiased estimator of Σ, $\overline{r}$ and $\widehat{\Delta}$ include respectively the time-series-means and sample covariance matrix corresponding to the right-hand-side portfolio returns.[1] Under (3.3), Q has the *Hotelling* $T^2(p,\ n-s-1)$ distribution or alternatively,

$$\frac{Q(n-s-p)}{p(n-s-1)} \sim F(p, n-s-p)\,. \tag{3.7}$$

As argued above [see also Stewart (1997)] Λ is related to the GRS criterion as follows:

$$\Lambda - 1 = \frac{Q}{n-s-1}\,. \tag{3.8}$$

We thus see that the GRS results obtains from Theorem 4 under the special case of normal errors.

Zhou (1993) considers GRS's problem in models with elliptical distributions; the multivariate student t distribution is included as an example. In this context, the author demonstrates location/scale invariance of the GRS-type efficiency test statistic and exploits this property to derive simulation based p-values.[2] The "*Monte Carlo integration technique*" proposed to do this is highly related to MC tests. Although the method is presented, rather heuristically, as an exact procedure, the nuisance parameter problem is not completely dealt with by Zhou (1993). For instance, in the multivariate student t case, it is evident that the associated degrees-of-freedom do intervene in the null distribution of the test statistic. Location-Scale invariance yields pivotality (thus exactness) for known degrees-of-freedom. We reconsider this case formally in the next section.

Before we turn to other hypotheses, it is useful to point out an interesting diagnostic test from Zhou (1993). The author implements the same Monte Carlo integration technique proposed for the efficiency test, to obtain p-values for multivariate skewness and kurtosis tests. A note explains that the ensuing procedure is not *strictly exact* because it is residuals based. In other words, the author does not recognize that the invariance properties which were shown to hold in the case of the efficiency tests may also be exploited with multivariate normality tests. In the following section, we formally reconsider these tests, and show how to obtain relevant exact p-values. We also use the criteria to devise a confidence set for the intervening degrees of freedom under multivariate student t error distributions.

[1] MacKinlay (1987) proposes a similar statistic in the context of the single beta CAPM.

[2] See also Zhou (1991). In both articles, the LR statistic is expressed in terms of the roots of a determinantal equation that depends only on the constrained and unconstrained residuals cross-products. From there on, location-scale invariance is proved without the normal assumption. This is the same approach underlying Theorem 1, with the exception that the latter result is not restricted to efficiency tests.

3.2 Unknown risk-free rate

The econometric analysis is more complicated when the zero beta intercept is unknown and must be inferred using the return data [see, for example Gibbons (1982)]. In this case, efficiency is usually tested in the context of a total-returns CAPM based on (2.1) with $Y = [R_1, \ldots, R_p]$, $X = [\iota_n, \widetilde{R}_1, \ldots, \widetilde{R}_s]$ where $R_j = (R_{1j}, \ldots, R_{nj})'$ and $\widetilde{R}_k = (\widetilde{R}_{1k}, \ldots, \widetilde{R}_{nk})'$:

$$R_{ij} = b_{0j} + \textstyle\sum_{k=1}^{s} b_{kj}\widetilde{R}_{ik} + U_{ij}, \qquad i = 1, ..., n, \quad j = 1, ..., p \ . \tag{3.9}$$

The relevant null hypothesis is non-linear and takes the following form

$$H_{CAPM} : b_{0j} = \gamma \left(1 - \sum_{k=1}^{s} b_{kj}\right), \ j = 1, \ \ldots \ , \ p, \tag{3.10}$$

or alternatively

$$(1, \gamma, ..., \gamma)B = \gamma\iota_p'$$

where γ is the unknown zero-beta intercept and ι_p refers to a p-dimensional vector of ones.

Let $\mathfrak{L}$ denote the statistic from (2.5) to test (3.10):

$$LR = n\ln(\mathfrak{L}) \ , \quad \mathfrak{L} \ = |\widehat{\Sigma}_{CAPM}|/|\widehat{\Sigma}| \tag{3.11}$$

where $\widehat{\Sigma}_{CAPM}$ is the MLE under H_{CAPM} (3.10). For all error distribution under (2.2), the problem of testing H_{CAPM} satisfies the assumptions of Theorem 2, which implies the following exact distributional result.

Theorem 5 *Under* (3.9), (2.2) *and* (3.10), *the null distribution of Wilks' statistic* $\mathfrak{L}$ *defined by* (3.11) *may be bound as follows*

$$P[\mathfrak{L} \geq x] \leq P[\Lambda_* \geq x], \ \forall x,$$

where

$$\Lambda_* = |\widehat{\Sigma}_{02}|/|\widehat{\Sigma}| \ , \tag{3.12}$$

$\widehat{\Sigma}$ *is the unconstrained MLE,* $\widehat{\Sigma}_{02}$ *is the MLE under UL restrictions of the form*

$$H_{02} : b_{0j} = \gamma_0 \left(1 - \sum_{k=1}^{s} b_{kj}\right), \ j = 1, \ \ldots \ , \ p, \tag{3.13}$$

and γ_0 *is any known constant.*

Indeed, to implement Theorem 2, we need to come up with a UL hypothesis which is a special case of H_{CAPM} (3.10). When $\gamma = \gamma_0$ and γ_0 is known, (3.10) is UL. Furthermore, testing (3.10) where $\gamma = \gamma_0$ and γ_0 is known in the context of the total-returns market model corresponds to testing the GRS null hypothesis (3.3) in an excess-returns model where the risk-free rate is γ_0.

Now applying Theorems 4-5 and (2.10) leads to the following bound

$$P[((n-s-p)(\mathfrak{L}-1))/p \geq x] \leq P[F(p,n-s-p) \geq x], \ \forall x,$$

if the normality assumption is imposed. We will next use these results to study the bounds proposed by Shanken (1986) and Stewart (1997).

Shanken (1986) employs the statistic $Q(\widehat{\gamma})$, where, in the context of a total-returns CAPM,

$$Q(\gamma) = \frac{n\widehat{\alpha}'(\gamma)\widehat{S}^{-1}\widehat{\alpha}(\gamma)}{1+(\overline{R}-\gamma\iota_s)'\widehat{\Delta}^{-1}(\overline{R}-\gamma\iota_s)} ,$$

$$\widehat{\gamma} = \underset{\gamma}{\mathrm{ARGMIN}}\, Q(\gamma) \ , \ \widehat{\alpha}(\gamma) = \widehat{a} - \gamma(\iota_p - \widehat{B}_{(s)}\iota_s) \ , \tag{3.14}$$

$\widehat{a}$ is the p-dimensional vector of OLS intercept estimates, $\widehat{B}_{(s)}$ is the $(p \times s)$ submatrix of $\widehat{B}$, the OLS estimates matrix, which excludes intercepts, $\widehat{S}$ is the unbiased estimate of Σ, $\overline{R}$ and $\widehat{\Delta}$ include respectively the time-series-means and sample covariance matrix corresponding to the right-hand-side total-portfolio-returns. Shanken shows that:

1. the LR statistic for testing (3.10) is a transformation of $Q(\widehat{\gamma})$, specifically
$$LR = n\ln\left(1+\frac{Q(\widehat{\gamma})}{n-s-1}\right), \tag{3.15}$$
2. $\widehat{\gamma}$ is the constrained MLE of γ, and
3. the null distribution of $Q(\widehat{\gamma})$ may be bounded by the Hotelling $T^2(p, n-s-1)$ distribution, or alternatively $((n-s-p)Q(\widehat{\gamma}))\,/\,(p(n-s-1))$ can be bounded by the $F(p,n-s-p)$ distribution.

Independently, Stewart (1997) shows that the statistic $((n-s-p)(\mathfrak{L}-1))/p$ can be bounded by the $F(p,n-s-p)$ distribution. The latter results is obtained applying (2.10) to Dufour (1989)'s conservative bounds test procedure. From (3.11)-(3.15) it is evident that: (i) Shanken and Stewart's bounds are equivalent, and (ii) both results obtain from Theorem 5 under the special case of normal errors.

It is important to recall that Theorem 5 extends the Shanken-Stewart bounds beyond the Gaussian context. As described in section 2 (see Corollary 3), the corresponding BMC p-value may be easily obtained by simulation. For this problem, no other exact test seems available. Finally, note that in the BMC algorithm, one only needs to draw the matrix W. The choice for γ_0 is practically irrelevant. To understand this point, reconsider the Gaussian model. In this case, it is evident that the reasoning which relates the bound from Theorem 5 to the Shanken-Stewart bound does not depend on the value of γ_0. Indeed, the bounding distribution under normality is $F(p,n-s-p)$, $\forall\ \gamma_0$.

4 Efficiency tests: the case of multivariate-t error distributions with unknown degrees-of-freedom

In this section, we extend the above results to a distributional family of particular interest: the case of multivariate-t errors with unknown degrees-of-freedom. Specifically, we suppose that the rows of the error matrix are distributed independently as in (3.5). The proposed procedure applies to problems with observable or unknown risk-free rate.

At this stage, two points deserve notice. First, as noted above, for a given κ, the distributional hypothesis underlying (3.5) satisfies (2.2). Thus the MC p-values associated with both Theorems 4 and 5 is exact for given κ. Secondly, whether κ is viewed (from an empirical perspective) as a parameter of interest or a nuisance parameter, it is important, for the precision of the efficiency test, to devise a decision rule which takes κ explicitly into consideration. Otherwise, level control may not hold.

Here we propose a solution based on the finite sample test approach proposed by Dufour and Kiviet (1996). The method requires the application of two sequential techniques: (1) an exact confidence set for κ, and (2) the MC test technique presented above, maximized over-all values of κ in the latter confidence set. It is important to note that if an overall α-level test is desired, then the pre-test confidence set and the maximized Monte Carlo test should be applied with levels $(1-\alpha_1)$ and α_2, respectively, so that

$$\alpha = \alpha_1 + \alpha_2 .$$

In empirical application considered next, we use $\alpha_1 = \alpha_2 = \alpha/2$.

To set focus, let us refer to the $(1-\alpha_1)$ confidence set for κ as $\mathcal{C}(y)$ where y denotes the returns data (as in e.g. (2.3)). Since a procedure to derive such a confidence set is not available, we provide one in what follows. Observe that, in principle, $\mathcal{C}(y)$ needs not be bounded. We then present the maximized MC algorithm. Since the latter procedure is not specific to our proposed confidence set, our presentation will thus be expressed in terms of any valid $\mathcal{C}(y)$. For proofs and further references, see Dufour and Kiviet (1996) and Dufour (1995).

4.1 A confidence set for the degrees-of-freedom parameter

We now present an explicit set estimation method to obtain $\mathcal{C}(y)$, which builds on Zhou (1993). Using the multivariate skewness and kurtosis criteria applied by Zhou (1993), we first propose a formal test for $\kappa = \kappa_0$, where κ_0 is any known integer. The test is then "inverted" to obtain a confidence set for κ.

The skewness and kurtosis criteria in question are:

$$\text{sk} = \frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}\widehat{d}_{ij}^{3}, \tag{4.1}$$

$$\text{ku} = \frac{1}{T}\sum_{i=1}^{T}\widehat{d}_{ii}^{4}, \tag{4.2}$$

where $\widehat{d}_{ij}$ are the elements of the matrix

$$\widehat{D} = \widehat{U}(\widehat{U}'\widehat{U})^{-1}\widehat{U}', \qquad \widehat{U} = Y - X\widehat{B}.$$

The latter statistics where introduced by **?** in a location-scale model (the case where the regressor matrix reduces to a vector of ones) to assess deviations from multivariate normality. Zhou (1993) argues that these criteria may serve to test for departures from a multivariate t(κ)-distribution, if cut-off points are appropriately "approximated", e.g. by simulation, imposing t(κ) errors. In view of this, the author estimates the degrees-of-freedom as follows: a few values for κ are retained by trial-and-error techniques (no further details are provided); then (modified) skewness and kurtosis tests are applied which confirm that the values retained do not yield significant lack-of-fit.

To devise an inference procedure which formalizes the above technique, we first prove an important invariance property regarding residuals based skewness and kurtosis tests.

Proposition 6 *Under* (2.1), *and for all error distributions compatible with* (2.2), *the multivariate skewness and kurtosis criteria* (4.1) *and* (4.2) *are distributed, respectively, like* $\frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T} d_{ij}^3$ *and* $\frac{1}{T}\sum_{i=1}^{T} d_{ii}^4$, *where* d_{ij} *are the elements of the matrix* $MW\,(W'MW)^{-1}\,W'M$, $M = I - X(X'X)^{-1}\,X'$, W *is defined by* (2.2).

PROOF: On observing that

$$\widehat{U} = MU$$

it is straightforward to see that

$$\begin{aligned}
\widehat{D} &= MU(U'MU)^{-1}U'M \\
&= MU(J^{-1})'J'(U'MU)^{-1}J(J^{-1})U'M \\
&= MU(J^{-1})'\left((J^{-1})U'MU(J^{-1})'\right)^{-1}(J^{-1})U'M \\
&= MW\left(W'MW\right)^{-1}W'M.
\end{aligned}$$

Since (2.2) implies that W has a known distribution, it follows that $\widehat{D}$ (and consequently sk and ku) are nuisance-parameter invariant.[3] On the basis of this

[3] In the literature on multivariate normality tests, this property is recognized (under Gaussianity) in location-scale models. Here we show that nuisance parameter invariance holds even though residuals (rather then observables) are used to construct the tests, and for all error distributions which satisfy assumption (2.2). As noted above, Zhou (1993) does not formally recognize the pivotality property we exploit here, in the case of residuals based tests.

result, we propose the following skewness-and-kurtosis based statistics to test $\kappa = \kappa_0$:

$$\begin{aligned} \text{esk}(\kappa_0) &= \left|\text{sk} - \overline{\text{sk}}(\kappa_0)\right|, \quad (4.3)\\ \text{eku}(\kappa_0) &= \left|\text{ku} - \overline{\text{ku}}(\kappa_0)\right|, \quad (4.4)\end{aligned}$$

where $\overline{\text{sk}}(\kappa_0)$ and $\overline{\text{ku}}(\kappa_0)$ are simulation-based estimates of the expected sk and ku given $\text{t}(\kappa_0)$ errors. These may be obtained as follows, given κ_0.

A1. For a given number $\mathfrak{N}$ of replications, draw $W^j = [W_1^j, \ \ldots \ , W_p^j]$, $j = 1, ..., \mathfrak{N}$, conforming with (3.5).

A2. From each simulated error matrix W^j, compute

$$\widehat{D}_j = MW^j \left(W^{j\prime}MW^j\right)^{-1} W^{j\prime}M,$$

$j = 1 \ , \ \ldots \ , \ \mathfrak{N}$. These provide $\mathfrak{N}$ replications of sk and ku, applying (4.1) and (4.2), namely sk_j and ku_j.

A3. Then calculate

$$\overline{\text{sk}}(\kappa_0) = \textstyle\sum_{j=1}^{\mathfrak{N}} \text{sk}_j / \mathfrak{N}, \qquad \overline{\text{ku}}(\kappa_0) = \textstyle\sum_{j=1}^{\mathfrak{N}} \text{ku}_j / \mathfrak{N}$$

The question is of course how to obtain exact cut-off points for (4.3) and (4.4). As in sections 2-3, we suggest to apply a Monte Carlo test method, which may be run as follows, given κ_0.

B1. Let esk^0 and eku^0 denote the observed test statistics.

B2. For a given number $\mathcal{M}$ of replications, and independently from the simulation performed to obtain $\overline{\text{sk}}(\kappa_0)$ and $\overline{\text{ku}}(\kappa_0)$ (i.e. step A2. above), draw $W^m = [W_1^m, \ \ldots \ , W_p^m]$, $m = 1, ..., \mathcal{M}$, conforming with (3.5).

B3. From each simulated error matrix W^m, compute

$$\widehat{D}_m = MW^m \left(W^{m\prime}MW^m\right)^{-1} W^{m\prime}M|,$$

$j = 1 \ , \ \ldots \ , \ \mathcal{M}$. Conformably, derive, applying (4.1) and (4.2), $\mathcal{M}$ replications of sk and ku, sk_m and ku_m.

B4. Conditioning on $\overline{\text{sk}}(\kappa_0)$ and $\overline{\text{ku}}(\kappa_0)$ (generated only once as in steps (A1-A3)), obtain, applying (4.3) and (4.4), $\mathcal{M}$ replications of esk and eku, esk_m and eku_m.

B5. Obtain (respectively) the ranks $\widehat{R}_{\mathcal{M}}(\text{esk}_0)$ and $\widehat{R}_{\mathcal{M}}(\text{eku}_0)$ of esk_0 and eku_0 in the series

$$\{\ \text{esk}_0,\ \ \text{esk}_1,\ ...,\ \text{esk}_{\mathcal{M}}\ \}$$

and

$$\{\ \text{eku}_0,\ \ \text{eku}_1,\ ...,\ \text{eku}_{\mathcal{M}}\ \}.$$

Then MC p-values correspond to

$$\begin{aligned} \widehat{p}_{\mathcal{M}}(\text{esk}_0\,|\kappa_0) &= 1-\frac{\widehat{R}_{\mathcal{M}}(\text{esk}_0)-1}{\mathcal{M}+1}, \\ \widehat{p}_{\mathcal{M}}(\text{eku}_0\,|\kappa_0) &= 1-\frac{\widehat{R}_{\mathcal{M}}(\text{eku}_0)-1}{\mathcal{M}+1}, \end{aligned}$$

where the conditioning on κ_0 is emphasized for further reference.

It is important to note at this stage that although the same value $\overline{\text{sk}}(\kappa_0)$ and $\overline{\text{ku}}(\kappa_0)$ is used for all replications of the modified test statistics, the latter remain exchangeable, which provides, along with Proposition 1, the necessary conditions for the validity of the MC p-values in B5 ; see Dufour (1995).

The tests just outlined are interesting for their own right, since procedures to formally test $\kappa = \kappa_0$ seem lacking. Now to obtain the confidence set, we propose to "invert" the latter tests (jointly). Specifically, the confidence set we retain for κ corresponds to the values $1 \leq \kappa_0 \leq T-K-p$, where $\widehat{p}_{\mathcal{M}}(\text{eku}_0\,|\kappa_0) > \alpha_{11}$ and $\widehat{p}_{\mathcal{M}}(\text{esk}_0\,|\kappa_0) > \alpha_{12}$, and α_{11} and α_{12} are chosen such that $\alpha_{11}+\ \alpha_{12} = \alpha_1$.

Although concerns regarding the possible conservative character of the above inference procedure may not be ruled out, the confidence set is definitely an improvement over available trial and error methods. From the results reported below, we do observe that the estimated confidence set are quite wide, yet the associated efficiency test decision is not adversely affected.

Finally, note that improved (exact) versions of Mardia's multi-normality tests may also be obtained as outlined above if the underlying W matrices are drawn from a multivariate normal distribution. In the empirical application presented next, we perform these tests which we denote $\mathcal{S}_{\mathcal{K}}$ and $\mathcal{K}_{\mathcal{U}}$.

4.2 The maximized Monte Carlo p-value

Let us now present the maximized MC algorithm. $\forall\kappa \in \mathcal{C}(y)$, and applying Theorem 4 or 5 (depending on the hypothesis of interest), one may obtain the MC p-values $\widehat{p}_N(\Lambda_0|\kappa)$ or $\widehat{p}_N^*(\mathfrak{L}_0|\kappa)$ [see (2.9) and (2.16)]. For ease of presentation, let us refer to the p-value of interest $\widehat{p}_{MC}(\kappa)$. Let

$$Q_U(\kappa) = \sup_{\kappa\in\mathcal{C}(y)} \widehat{p}_{MC}(\kappa), \tag{4.5}$$

and then critical region

$$Q_U(\kappa) \leq \alpha_2 \tag{4.6}$$

is exactly of level $\alpha_1 + \alpha_2$.The associated test is conservative in the following sense: if indeed $Q_U(\kappa) \leq \alpha_2$ for the sample at hand, then the test is most certainly significant. For further reference, the values of $\kappa \in \mathcal{C}(y)$ where $\widehat{p}_{MC}(\kappa) = Q_U(\kappa)$ are denoted $\mathcal{C}^U(y)$.

The maximization problem underlying $Q_U(\kappa)$ may be computer intensive in general applications of this method. In this case however, κ takes only integer values and the statistic underlying the p-value are straightforward to simulate.[4] To conclude, we re-iterate the arguments from (Dufour (1989)), (Dufour (1997)) and Dufour and Khalaf (1998) on the conservative character of such bounds tests. Since the bound is so easy to use, we recommend to implement it in conjunction with, and not necessarily as an alternative to, e.g. bootstrap or Maximized Monte Carlo tests. Indeed, a significant bounds test is compelling. Thus, one may run the bounds procedure first then proceed if necessary to alternative tests.

5 Empirical application

To illustrate the above results, we present an empirical application on a CAPM test with an unknown risk-free rate. Conforming with the notation set above, we use a total-returns CAPM as defined in section 3.2 with a single benchmark portfolio. The model at hand is (2.1) with $Y = [R_1, \ldots, R_p]$, $X = [\iota_n, \widetilde{R}_M]$ where R_j are the vectors of portfolio returns under consideration and $\widetilde{R}_M$ is the vector of returns on the market benchmark. The relevant null hypothesis is

$$(1, \gamma)B = \gamma\iota_p' \tag{5.7}$$

or alternatively

$$b_{0j} = \gamma(1 - b_{1j}), \; j = 1, \; \ldots \; , \; p.$$

We use nominal monthly returns over the period going from January 1926 to December 1995, obtained from the University of Chicago's Center for Research in Security Prices (CRSP). As in **?**, our data includes 12 portfolios of New York Stock Exchange (NYSE) firms grouped by standard two-digit industrial classification (SIC). Real returns are computed using the consumer price index. Table 1 provides a list of the different sectors used as well as the SIC codes included in the analysis. For each month the industry portfolios are comprised of those firms for which the return, price per common share and number of shares outstanding are recorded by CRSP. Furthermore, the portfolios are value-weighted in each month. In order

[4]In the case of H_{CAPM} (3.10) an alternative exact test (which might be "less" conservative) is the maximized Monte Carlo test based on $\mathfrak{L}$ (rather than its bound) as proposed by (Dufour (1995)). However, the relevant nuisance parameter list in this case will not be restricted to κ, which may complicate the maximization problem importantly.

to test the CAPM, we proxy the market return with the value-weighted NYSE return, also available from CRSP.

¿From the results in Zhou (1991), note that the LR statistic may be obtained analytically as

$$LR = -T \ln(\lambda)$$

where λ is the smallest root of the determinantal equation

$$|X'\widetilde{Y}(\widetilde{Y}'\widetilde{Y})^{-1}\widetilde{Y}'X - \lambda X'X| = 0,$$

and $\widetilde{Y} = Y - \iota_p' \otimes \widetilde{R}_M$.

Our results are summarized in tables 2.1 and 2.2. The latter excludes January returns and October 1987. As is usual in this literature, we estimate and test the model over 5-years sub-samples. We report: the p-values of the modified multi-normality tests $\mathcal{S}_{\mathcal{K}}$ and $\mathcal{K}_{\mathcal{U}}$ (see section 4) , LR and its asymptotic $\chi^2(p-1)$ p-value, the Gaussian based and the largest student-t based bounds MC p-value associated with LR (respectively, $p_{\mathcal{N}}$ and Q_U), the confidence set for κ ($\mathcal{C}(y)$), and its subset which maximizes the bounds p-values (denoted $\mathcal{C}^U(y)$). Figures 1-14 illustrate how the bounds p-value varies overall $\mathcal{C}(y)$. Several points are worth noting:

- Normality is rejected in many sample subsets
- Asymptotic p-values are spuriously significant quite often (e.g. 1941-55).
- The maximal p-values exceed the Gaussian-based p-value. It is relatively "easier" to reject the efficiency hypothesis under normality. Conversely, recall that the Gaussian model obtains with $\kappa \to \infty$. So if $p_{\mathcal{N}}$ exceeds the significance level, then the largest p-value, *a fortiori*, also exceeds the significance level. Then the decision implied by a non-significant Gaussian p-value is exactly conclusive (i.e. there is no need to re-consider t-based p-values if $p_{\mathcal{N}}$ fails to reject).
- Although $\mathcal{C}(y)$ is quite wide, it is evident from Figures 1-14 that restricting this set further does not change the tests decision importantly. Specifically, the p-values do not seems to fluctuate a lot throughout $\mathcal{C}(y)$, at least in this application.
- In spite of the 2-step decision rule and the conservative pre-test which served to obtain $\mathcal{C}(y)$, we still observe rejections in some sub-samples.

6 Conclusion

We have shown that in Gaussian or non-Gaussian contexts, the exact test procedure proposed by Dufour and Khalaf (1998) may be used to perform portfolio efficiency tests. Two earlier (bounds) efficiency tests were also shown to be a special case of the latter procedure, which also provided a proof of their equivalence. We have specifically illustrated how to deal exactly with student-t errors.

To conclude, it is worth emphasizing that the procedure proposed in by Dufour and Khalaf (1998) is not restricted to efficiency tests. In published empirical multivariate regressions based studies, interest has typically centered on asymptotic Wald-type tests. In view of the well known problems associated with such tests, reliance on asymptotics is not surprising in the absence of (applicable) exact results. The bound discussed in this paper provide a very useful approach to conducting exact tests in multivariate asset pricing models.

Table 1: Portfolio definitions

Portfolio number	Industry Name	Two-digit SIC codes
1	Petroleum	13, 29
2	Finance and real estate	60-69
3	Consumer durables	25, 30, 36, 37, 50, 55, 57
4	Basic industries	10, 12, 14, 24, 26, 28, 33
5	Food and tobacco	1, 20, 21, 54
6	Construction	15-17, 32, 52
7	Capital goods	34, 35, 38
8	Transportation	40-42, 44, 45, 47
9	Utilities	46, 48, 49
10	Textile and trade	22, 23, 31, 51, 53, 56, 59
11	Services	72, 73, 75, 80, 82, 89
12	Leisure	27, 58, 70, 78, 79

This table presents portfolios according to their number and sector as well as the SIC codes included in each portfolio using the same classification as **?**.

Table 2.1 Efficiency tests including January returns[5]

	Normality		Efficiency					
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Sample	$\mathcal{S}_{\mathcal{K}}$	$\mathcal{K}_{\mathcal{U}}$	LR	p_∞	$p_{\mathcal{N}}$	Q_U	$\mathcal{C}(y)$	$\mathcal{C}^U(y)$
1/31/27-12/31/30	.001	.001	11.696	.2309	.415	.427	2-20	11,15,16
1/31/31-12/31/35	.001	.001	6.441	.6951	.858	.863	2-8	8
1/31/36-12/31/40	.001	.001	7.318	.6041	.773	.779	3-18	9,12
1/31/41-12/31/45	.001	.001	17.054	.0478	.133	.133	2-14	3
1/31/46-12/30/50	.001	.001	20.546	.0148	.060	.068	3-16	11
1/31/51-12/30/55	.002	.002	15.476	.0787	.189	.196	≥ 3	9
1/31/56-12/30/60	.127	.240	21.700	.0099	.046	.047	≥ 3	10
1/31/61-12/31/65	.762	.484	24.060	.0042	.014	.023	≥ 3	3
1/31/66-12/31/70	.112	.012	13.836	.1283	.301	.311	≥ 3	27
1/29/71-12/31/75	.001	.001	8.043	.5298	.713	.719	3-29	20
1/30/76-12/31/80	.001	.001	23.653	.0049	.021	.032	2-16	4
1/30/81-12/31/85	.003	.003	12.820	.1709	.368	.381	≥ 3	21
1/31/86-12/31/90	.041	.019	28.933	.0007	.003	.020	≥ 3	3
1/31/91-12/29/95	.302	.103	4.850	.8472	.932	.939	≥ 3	22

Table 2.2 Efficiency tests excluding January returns (and October 1997)

	Normality		Efficiency Tests					
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Sample	$\mathcal{S}_{\mathcal{K}}$	$\mathcal{K}_{\mathcal{U}}$	LR	p_∞	$p_{\mathcal{N}}$	Q_U	$\mathcal{C}(y)$	$\mathcal{C}^U(y)$
1/31/27-12/31/30	.001	.001	10.357	.3224	.558	.561	2-21	11,18
1/31/31-12/31/35	.001	.001	6.865	.6512	.798	.796	2-8	7
1/31/36-12/31/40	.001	.001	11.355	.2522	.465	.477	3-20	15
1/31/41-12/31/45	.001	.002	16.858	.0510	.123	.131	3-30	23
1/31/46-12/30/50	.001	.001	26.842	.0015	.009	.014	2-25	18
1/31/51-12/30/55	.003	.004	15.790	.0714	.187	.203	3-40	21
1/31/56-12/30/60	.273	.502	26.148	.0019	.006	.011	≥ 3	16
1/31/61-12/31/65	.975	.357	25.579	.0016	.004	.011	≥ 3	4,8
1/31/66-12/31/70	.520	.100	11.081	.2702	.471	.481	≥ 3	31,43
1/29/71-12/31/75	.001	.003	13.591	.1376	.293	.308	3-21	5
1/30/76-12/31/80	.001	.001	20.683	.0141	.081	.088	2-19	2
1/30/81-12/31/85	.003	.005	14.165	.1166	.263	.279	≥ 3	34
1/31/86-12/31/90	.033	.021	32.855	.0001	.001	.004	≥ 3	3
1/31/91-12/29/95	.410	.316	5.324	.8052	.927	.931	≥ 4	15

[5]Numbers in (1)-(2), (4)-(6) are p-values pertaining to: the modified multi-normality tests [(1)-(2)], LR's χ^2, the Gaussian and the largest student-t based bounds MC p-value [(4)-(6)]. (3) presents LR, (7)-(8) present the degrees-of-freedom confidence set, and its subset which maximizes the bounds p-values.

References

Campbell, Y. Y., Lo, A. W. and MacKinlay, A. C. (1997), *The Econometrics of Financial Markets*, Princeton University Press, New Jersey.

Dufour, J.-M. (1989), 'Nonlinear hypotheses, inequality restrictions, and non-nested hypotheses: Exact simultaneous tests in linear regressions', *Econometrica* **57**, 335–355.

Dufour, J.-M. (1995), Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics in econometrics, Technical report, C.R.D.E., Université de Montréal.

Dufour, J.-M. (1997), 'Some impossibility theorems in econometrics, with applications to structural and dynamic models', *Econometrica* **65**, 1365–1389.

Dufour, J.-M. and Khalaf, L. (1998), Simulation based finite and large sample inference methods in multivariate regressions and seemingly unrelated regressions, Technical report, C.R.D.E., Université de Montréal. 36 pages.

Dufour, J.-M. and Khalaf, L. (1999), Monte Carlo test methods in econometrics, Technical report, C.R.D.E., Université de Montréal, and GREEN, Université Laval. 61 pages.

Dufour, J.-M. and Kiviet, J. F. (1996), 'Exact tests for structural change in first-order dynamic models', *Journal of Econometrics* **70**, 39–68.

Gibbons, M. R. (1982), 'Multivariate tests of financial models: A new approach', *Journal of Financial Economics* **10**, 3–27.

Gibbons, M. R., Ross, S. A. and Shanken, J. (1989), 'A test of the efficiency of a given portfolio', *Econometrica* **57**, 1121–1152.

MacKinlay, A. C. (1987), 'On multivariate tests of the Capital Asset Pricing Model', *Journal of Financial Economics* **18**, 341–372.

Rao, C. R. (1973), *Linear Statistical Inference and its Applications*, second edn, John Wiley & Sons, New York.

Shanken, J. (1986), 'Testing portfolio efficiency when the zero-beta rate is unknown: A note', *Journal of Finance* **41**, 269–276.

Stewart, K. G. (1997), 'Exact testing in multivariate regression', *Econometric Reviews* **16**, 321–352.

Zhou, G. (1991), 'Small sample tests of portfolio efficiency', *Journal of Financial Economics* **30**, 165–191.

Zhou, G. (1993), 'Asset-pricing tests under alternative distributions', *The Journal of Finance* **48**, 1927–1942.