Finding small stabilizers for unstable graphs

Laura Sanità

Combinatorics and Optimization Department

University of Waterloo

Joint work with:

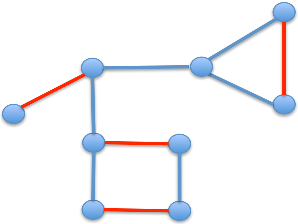
A. Bock, K. Chandrasekaran, J. Könemann, B.Peis

Matching and Stable Graphs

• A matching of a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ is incident into at most one edge of M

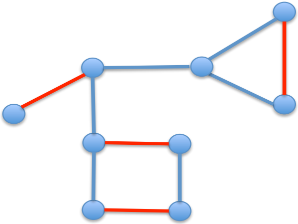
Matching and Stable Graphs

- A matching of a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ is incident into at most one edge of M



Matching and Stable Graphs

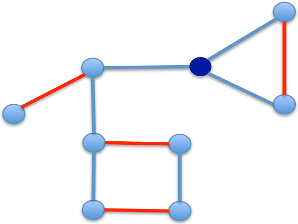
• A matching of a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ is incident into at most one edge of M



• A vertex $v \in V$ is called inessential if there exists a matching in G of maximum cardinality that exposes v.

Matching and Stable Graphs

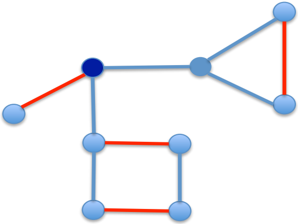
• A matching of a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ is incident into at most one edge of M



• A vertex $v \in V$ is called inessential if there exists a matching in G of maximum cardinality that exposes v.

Matching and Stable Graphs

• A matching of a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ is incident into at most one edge of M



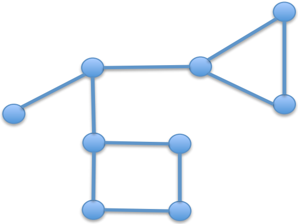
• A vertex $v \in V$ is called essential if there is no matching in G of maximum cardinality that exposes v.

Matching and Stable Graphs

- G is said to be stable if the set of its inessential vertices forms a stable set.

Matching and Stable Graphs

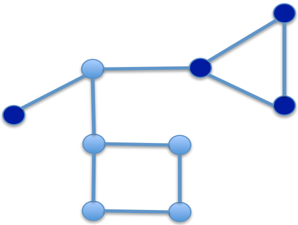
- G is said to be stable if the set of its inessential vertices forms a stable set.



- Unstable Graph

Matching and Stable Graphs

- G is said to be stable if the set of its inessential vertices forms a stable set.



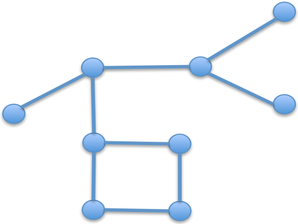
- Unstable Graph

Matching and Stable Graphs

- G is said to be stable if the set of its inessential vertices forms a stable set.

Matching and Stable Graphs

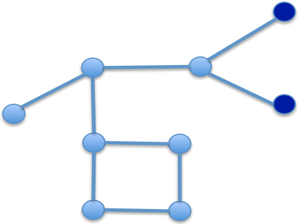
- G is said to be stable if the set of its inessential vertices forms a stable set.



- Stable Graph

Matching and Stable Graphs

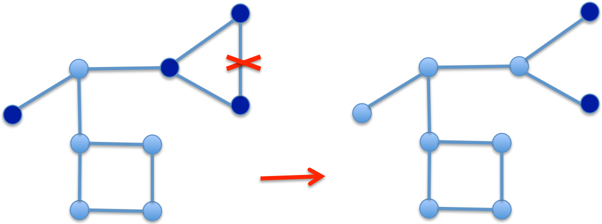
- G is said to be stable if the set of its inessential vertices forms a stable set.



- Stable Graph

Stabilizers

- A stabilizer for an unstable graph G is a subset $F \subseteq E$ s.t. $G \setminus F$ is stable.



Finding small stabilizers

Finding small stabilizers

- In this talk we focus on the following optimization problem:

Finding small stabilizers

- In this talk we focus on the following optimization problem:

Given an unstable graph G, find a stabilizer of minimum cardinality.

Finding small stabilizers

- In this talk we focus on the following optimization problem:

Given an unstable graph G, find a stabilizer of minimum cardinality.

- A recent motivation to study this problem comes from the theory of network bargaining games

Network Bargaining Games

Network Bargaining Games

• [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where

Network Bargaining Games

• [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where

- Vertices represent players
- Edges represent potential deals between players

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where
  - Vertices represent players
  - Edges represent potential deals between players
- Players can enter in a deal with at most one neighbour

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where
  - Vertices represent players
  - Edges represent potential deals between players
- Players can enter in a deal with at most one neighbour

$$\rightarrow \text{matching } M$$

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where
  - Vertices represent players
  - Edges represent potential deals between players
- Players can enter in a deal with at most one neighbour

$$\rightarrow \text{matching } M$$

- If players u and v make a deal, they agree on how to split a unit value

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where
  - Vertices represent players
  - Edges represent potential deals between players

- Players can enter in a deal with at most one neighbour

$$\rightarrow \text{matching } M$$

- If players u and v make a deal, they agree on how to split a unit value

$$\rightarrow \text{allocation } y \in \mathbb{R}^V:$$
$$y_u + y_v = 1 \text{ for all } \{uv\} \in M$$
$$y_u = 0 \text{ if } u \text{ is exposed by } M.$$

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] recently introduced a *network bargaining game* described by a graph $G = (V, E)$ where
  - Vertices represent players
  - Edges represent potential deals between players

- Players can enter in a deal with at most one neighbour

$$\rightarrow \text{matching } M$$

- If players u and v make a deal, they agree on how to split a unit value

$$\rightarrow \text{allocation } y \in \mathbb{R}^V:$$
$$y_u + y_v = 1 \text{ for all } \{uv\} \in M$$
$$y_u = 0 \text{ if } u \text{ is exposed by } M.$$

- An outcome for the game is a pair (M, y)

Network Bargaining Games

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \; : \; \{uv\} \in \delta(u) \setminus M\}$$

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \ : \ \{uv\} \in \delta(u) \setminus M\}$$

- If $\alpha_u > y_u \ \Rightarrow$ there exists a neighbour v of u with $1 - y_v > y_u$

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \ : \ \{uv\} \in \delta(u) \setminus M\}$$

- If $\alpha_u > y_u \Rightarrow$ there exists a neighbour v of u with $1 - y_v > y_u$

  $\rightarrow$ player u has an incentive to enter in a deal with v!

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \ : \ \{uv\} \in \delta(u) \setminus M\}$$

- If $\alpha_u > y_u \ \Rightarrow$ there exists a neighbour v of u with $1 - y_v > y_u$

  $\rightarrow$ player u has an incentive to enter in a deal with v!

- An outcome (M, y) is stable if $y_u + y_v \geq 1$ for all edges $\{uv\} \in E$.

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \ : \ \{uv\} \in \delta(u) \setminus M\}$$

- If $\alpha_u > y_u \ \Rightarrow$ there exists a neighbour v of u with $1 - y_v > y_u$

  $\rightarrow$ player u has an incentive to enter in a deal with v!

- An outcome (M, y) is stable if $y_u + y_v \geq 1$ for all edges $\{uv\} \in E$.

  $\rightarrow$ no player has an incentive to deviate

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \ : \ \{uv\} \in \delta(u) \setminus M\}$$

- If $\alpha_u > y_u \ \Rightarrow$ there exists a neighbour v of u with $1 - y_v > y_u$

  $\rightarrow$ player u has an incentive to enter in a deal with v!

- An outcome (M, y) is stable if $y_u + y_v \geq 1$ for all edges $\{uv\} \in E$.

  $\rightarrow$ no player has an incentive to deviate

- A stable outcome (M, y) is balanced if $y_u - \alpha_u = y_v - \alpha_v$ for all $\{uv\} \in M$

Network Bargaining Games

- For a given outcome (M, y) player u gets an outside alternative

$$\alpha_u := \max\{1 - y_v \ : \ \{uv\} \in \delta(u) \setminus M\}$$

- If $\alpha_u > y_u \ \Rightarrow$ there exists a neighbour v of u with $1 - y_v > y_u$

$\rightarrow$ player u has an incentive to enter in a deal with v!

- An outcome (M, y) is stable if $y_u + y_v \geq 1$ for all edges $\{uv\} \in E$.

$\rightarrow$ no player has an incentive to deviate

- A stable outcome (M, y) is balanced if $y_u - \alpha_u = y_v - \alpha_v$ for all $\{uv\} \in M$

$\rightarrow$ the values are "fairly" split among the players

Network Bargaining Games

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] proved that

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] proved that

a balanced outcome exists

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] proved that

a balanced outcome exists

$\Leftrightarrow$

a stable one exists

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] proved that

a balanced outcome exists

$\Leftrightarrow$

a stable one exists

$\Leftrightarrow$

the correspondent graph G is stable.

Network Bargaining Games

- [Kleinberg & Tardos STOC'08] proved that

a balanced outcome exists

$\Leftrightarrow$

a stable one exists

$\Leftrightarrow$

the correspondent graph G is stable.

Question: Can we stabilize unstable games through minimal changes in the underlying network?

e.g. by *blocking* some potential deals?

[Biró, Kern & Paulusma, 2010, Könemann, Larson & Steiner, 2012]

Network Bargaining Games

• [Kleinberg & Tardos STOC'08] proved that

a balanced outcome exists
$\Leftrightarrow$
a stable one exists
$\Leftrightarrow$
the correspondent graph G is stable.

Question: Can we stabilize unstable games through minimal changes in the underlying network?
e.g. by *blocking* some potential deals?
[Biró, Kern & Paulusma, 2010, Könemann, Larson & Steiner, 2012]

• The combinatorial question behind it turns out to be exactly how to find small stabilizers for unstable graphs!

Our Main Results

Our Main Results

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Our Main Results

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

- Network Bargaining Interpretation: there is always a way to stabilize the game that

Our Main Results

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

• Network Bargaining Interpretation: there is always a way to stabilize the game that

- blocks min number of potential deals, and

Our Main Results

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

• Network Bargaining Interpretation: there is always a way to stabilize the game that

- blocks min number of potential deals, and
- does not decrease the total value the players can get!

Our Main Results

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

• Network Bargaining Interpretation: there is always a way to stabilize the game that

- blocks min number of potential deals, and
- does not decrease the total value the players can get!

Thm: Finding a minimum stabilizer is NP-Hard. Assuming UGC, it is hard to approximate within any factor better than 2.

Our Main Results

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

• Network Bargaining Interpretation: there is always a way to stabilize the game that

- blocks min number of potential deals, and
- does not decrease the total value the players can get!

Thm: Finding a minimum stabilizer is NP-Hard. Assuming UGC, it is hard to approximate within any factor better than 2.

Thm: There is a 4ω-approximation algorithm for general graphs, where ω is the sparsity of the graph.

Stable Graphs via LP

Stable Graphs via LP

- Stable graphs be characterized in terms of *fractional matchings and covers*.

Stable Graphs via LP

- Stable graphs be characterized in terms of *fractional matchings and covers*.

Def. a vector $x \in \mathbb{R}^E$ is a fractional matching if it is a feasible solution to (P):

$$\max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \ \forall v \in V, \ x \geq 0\} \qquad (P)$$

Stable Graphs via LP

- Stable graphs be characterized in terms of *fractional matchings and covers.*

Def. a vector $x \in \mathbb{R}^E$ is a fractional matching if it is a feasible solution to (P):

$$\max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \; \forall v \in V, \; x \geq 0\} \qquad (P)$$

Def. a vector $y \in \mathbb{R}^V$ is called a fractional vertex cover if it is a feasible solution to the dual (D) of (P):

$$\min\{\mathbf{1}^T y : y_u + y_v \geq 1 \; \forall \{uv\} \in E, \; y \geq 0\} \qquad (D)$$

Stable Graphs via LP

- Stable graphs be characterized in terms of *fractional matchings and covers.*

Def. a vector $x \in \mathbb{R}^E$ is a fractional matching if it is a feasible solution to (P):

$$\max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \ \forall v \in V, \ x \geq 0\} \qquad (P)$$

Def. a vector $y \in \mathbb{R}^V$ is called a fractional vertex cover if it is a feasible solution to the dual (D) of (P):

$$\min\{\mathbf{1}^T y : y_u + y_v \geq 1 \ \forall \{uv\} \in E, \ y \geq 0\} \qquad (D)$$

- **By duality:** size of a fractional matching $\leq$ size of a fractional vertex cover

Stable Graphs via LP

- Stable graphs be characterized in terms of *fractional matchings and covers.*

Def. a vector $x \in \mathbb{R}^E$ is a fractional matching if it is a feasible solution to (P):

$$\max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \ \forall v \in V, \ x \geq 0\} \qquad (P)$$

Def. a vector $y \in \mathbb{R}^V$ is called a fractional vertex cover if it is a feasible solution to the dual (D) of (P):

$$\min\{\mathbf{1}^T y : y_u + y_v \geq 1 \ \forall \{uv\} \in E, \ y \geq 0\} \qquad (D)$$

- **By duality:** size of a fractional matching $\leq$ size of a fractional vertex cover
Moreover, optimum value of (P) equals optimum value of (D)

Stable Graphs via LP

Proposition: G is stable if and only if the *cardinality of a maximum matching* $\nu(G)$ of G is equal to optimum value $\nu_f(G)$ of (P) and (D).

Stable Graphs via LP

Proposition: G is stable if and only if the *cardinality of a maximum matching* $\nu(G)$ of G is equal to optimum value $\nu_f(G)$ of (P) and (D).

(It follows from classical results e.g. [Uhry'75, Balas'81, Pulleyblank'87])

Stable Graphs via LP

Proposition: G is stable if and only if the *cardinality of a maximum matching* $\nu(G)$ of G is equal to optimum value $\nu_f(G)$ of (P) and (D).

(It follows from classical results e.g. [Uhry'75, Balas'81, Pulleyblank'87])

- In other words, G is stable if and only if

  cardinality of a max matching $=$ min size of a fractional vertex cover y.

Stable Graphs via LP

Proposition: G is stable if and only if the *cardinality of a maximum matching* $\nu(G)$ of G is equal to optimum value $\nu_f(G)$ of (P) and (D).

(It follows from classical results e.g. [Uhry'75, Balas'81, Pulleyblank'87])

- In other words, G is stable if and only if

  cardinality of a max matching = min size of a fractional vertex cover y.

- **Note:** such y does not necessarily have integer coordinates!

Stable Graphs via LP

Proposition: G is stable if and only if the *cardinality of a maximum matching* $\nu(G)$ of G is equal to optimum value $\nu_f(G)$ of (P) and (D).

(It follows from classical results e.g. [Uhry'75, Balas'81, Pulleyblank'87])

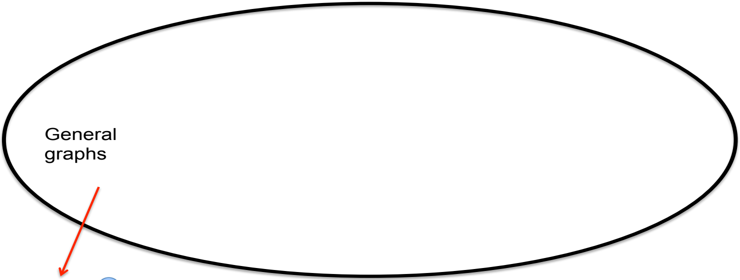
- In other words, G is stable if and only if

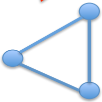
  cardinality of a max matching $=$ min size of a fractional vertex cover y.

- **Note:** such y does not necessarily have integer coordinates!

- A graph where the cardinality of a maximum matching $\nu(G)$ equals min size of an integral vertex cover is called a König-Egervary graph

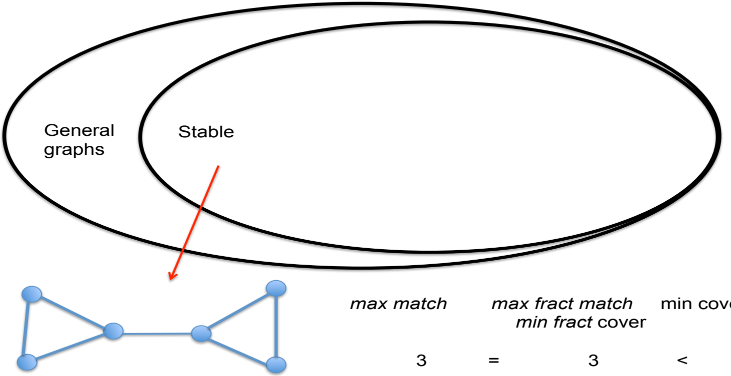
Stable Graphs via LP





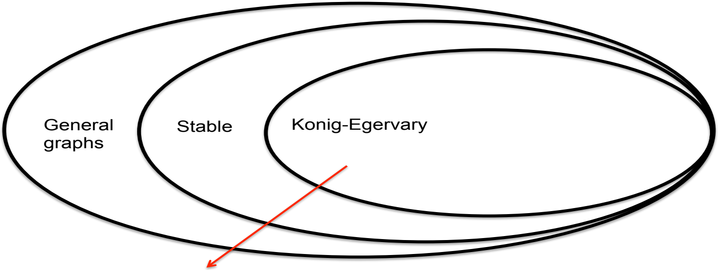
max match		*max fract match* - *min fract* cover				min cover
1	<	1.5	=	1.5	<	2

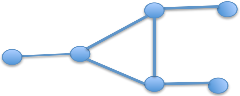
Stable Graphs via LP



max match		*max fract match* *min fract cover*		min cover
3	=	3	<	4

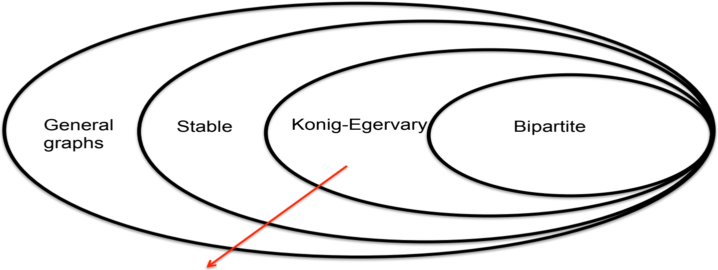
Stable Graphs via LP

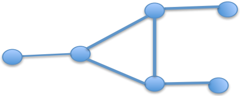




max match		*max fract match* *min fract cover*		min cover
3	=	3	=	3

Stable Graphs via LP





max match		*max fract match* *min fract cover*		min cover
3	=	3	=	3

Stable Graphs via LP

- As we showed

  Stable graphs $\supset$ König-Egervary graphs $\supset$ Bipartite graphs.

Stable Graphs via LP

- As we showed

$$\text{Stable graphs} \supset \text{König-Egervary graphs} \supset \text{Bipartite graphs.}$$

- All these classes are widely studied

Stable Graphs via LP

- As we showed

$$\text{Stable graphs} \supset \text{König-Egervary graphs} \supset \text{Bipartite graphs.}$$

- All these classes are widely studied but almost no algorithmic results are known for making a graph stable!

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

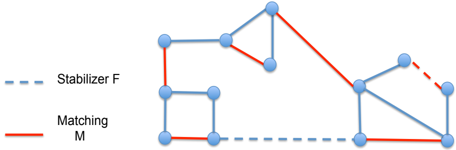
Proof: • Let M be a max matching with $|F \cap M|$ minimum.

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.

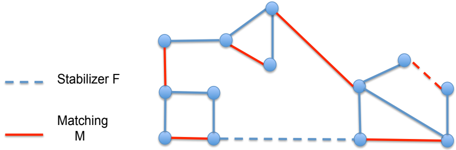


Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



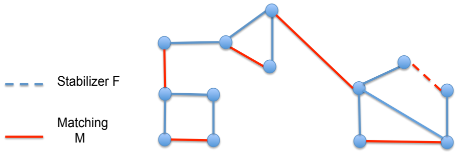
• Consider $G' := G \setminus (F \setminus M)$

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



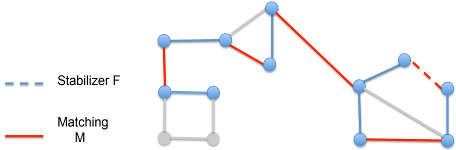
• Consider $G' := G \setminus (F \setminus M)$

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



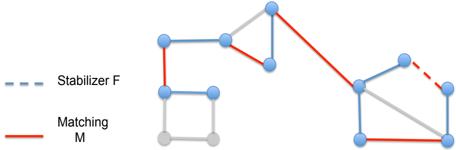
• Consider $G' := G \setminus (F \setminus M) \rightarrow$ There is a M-flower

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



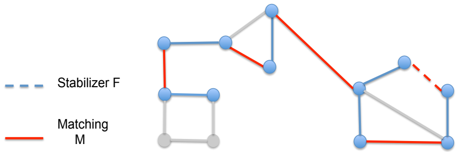
- Consider $G' := G \setminus (F \setminus M) \to$ There is a M-flower
- If it contains an edge from $F \to$ we could switch along an M-alternating path

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



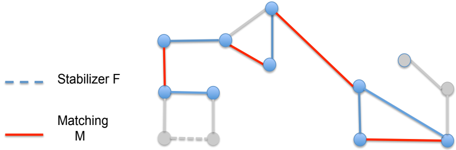
- Consider $G' := G \setminus (F \setminus M) \rightarrow$ There is a M-flower
- If it contains an edge from $F \rightarrow$ we could switch along an M-alternating path
- Contradiction!

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



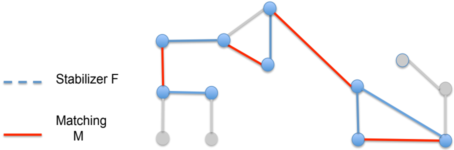
• Consider $G' := G \setminus (F \setminus M) \rightarrow$ There is a M-flower disjoint from F

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



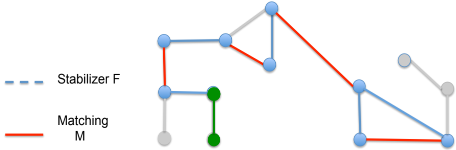
- Consider $G' := G \setminus (F \setminus M) \rightarrow$ There is a M-flower disjoint from F
- $M \setminus F$ is not maximum in $G \setminus F$

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



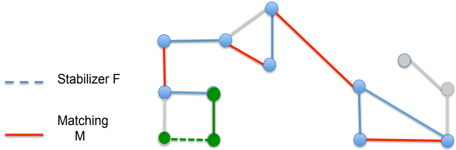
- Consider $G' := G \setminus (F \setminus M) \to$ There is a M-flower disjoint from F
- $M \setminus F$ is not maximum in $G \setminus F \to$ find a $(M \setminus F)$-augmenting path

Stabilizers and maximum Matchings

Thm: For a minimum stabilizer F of G we have

$$\nu(G \setminus F) = \nu(G)$$

Proof: • Let M be a max matching with $|F \cap M|$ minimum.



- Consider $G' := G \setminus (F \setminus M) \rightarrow$ There is a M-flower disjoint from F
- $M \setminus F$ is not maximum in $G \setminus F \rightarrow$ find a $(M \setminus F)$-augmenting path
- $\rightarrow$ implies existence of an even M-alternating path in G (Contradiction!) □

How do we find F^*?

How do we find $\mathbf{F}^*$?

- Previous theorem implies $G \setminus F^*$ contains a maximum matching M of G

How do we find F^*?

- Previous theorem implies $G \setminus F^*$ contains a maximum matching M of G

Easy Assumption: Suppose such matching M is given

How do we find $\mathbf{F}^*$?

- Previous theorem implies $G \setminus F^*$ contains a maximum matching M of G

Easy Assumption: Suppose such matching M is given

- We call $F \subseteq E$ an M-stabilizer if
  - F is a stabilizer

How do we find $\mathbf{F}^*$?

- Previous theorem implies $G \setminus F^*$ contains a maximum matching M of G

Easy Assumption: Suppose such matching M is given

- We call $F \subseteq E$ an M-stabilizer if
  - F is a stabilizer
  - $|F \cap M| = \emptyset$.

How do we find $\mathbf{F}^*$?

• Previous theorem implies $G \setminus F^*$ contains a maximum matching M of G

Easy Assumption: Suppose such matching M is given

• We call $F \subseteq E$ an M-stabilizer if

- F is a stabilizer
- $|F \cap M| = \emptyset$.

• Let us focus on the M-stabilizer problem, that is finding an M-stabilizer of minimum cardinality

How do we find $\mathbf{F}^*$?

- Previous theorem implies $G \setminus F^*$ contains a maximum matching M of G

Easy Assumption: Suppose such matching M is given

- We call $F \subseteq E$ an M-stabilizer if
  - F is a stabilizer
  - $|F \cap M| = \emptyset$.

- Let us focus on the M-stabilizer problem, that is finding an M-stabilizer of minimum cardinality

- How difficult is it?

When M is given

When M is given

Thm: The M-stabilizer problem is NP-hard, and no $(2 - \varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

When M is given

Thm: The M-stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture. Furthermore, the M-stabilizer problem admits a 2-approximation algorithm.

When M is given

Thm: The M-stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture. Furthermore, the M-stabilizer problem admits a 2-approximation algorithm.

- Approximation result is LP-based.

$$
\begin{aligned}
\min \quad & \sum_{\{uv\} \in E \setminus M} z_{uv} \\
\text{s.t.} \quad & y_u + y_v = 1 \quad \forall \{u, v\} \in M \\
& y_u + y_v + z_{uv} \geq 1 \quad \forall \{u, v\} \in E \setminus M \text{ and } u, v \text{ matched} \\
& y_v + z_{uv} \geq 1 \quad \forall \{u, v\} \in E \setminus M \text{ and } u \text{ unmatched} \\
& y \geq 0 \\
& z \geq 0 \text{ integer}
\end{aligned}
$$

When M is given

Thm: The M-stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture. Furthermore, the M-stabilizer problem admits a 2-approximation algorithm.

- Approximation result is LP-based.

$$
\begin{aligned}
\min \quad & \sum_{\{uv\} \in E \setminus M} z_{uv} \\
\text{s.t.} \quad & y_u + y_v = 1 \quad \forall \{u, v\} \in M \\
& y_u + y_v + z_{uv} \geq 1 \quad \forall \{u, v\} \in E \setminus M \text{ and } u, v \text{ matched} \\
& y_v + z_{uv} \geq 1 \quad \forall \{u, v\} \in E \setminus M \text{ and } u \text{ unmatched} \\
& y \geq 0 \\
& z \geq 0
\end{aligned}
$$

- **Main observation:** There always exists an optimal solution to the above LP that is half integral!

When M is NOT given

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2 - \varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

- From previous discussion, the difficulty lies in understanding which matching "survives" in $G \setminus F$!

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

- From previous discussion, the difficulty lies in understanding which matching "survives" in $G \setminus F$!

- Take an arbitrary max matching.

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

- From previous discussion, the difficulty lies in understanding which matching "survives" in $G \setminus F$!

- Take an arbitrary max matching. How bad can this choice be?

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

- From previous discussion, the difficulty lies in understanding which matching "survives" in $G \setminus F$!

- Take an arbitrary max matching. How bad can this choice be?

- Unfortunately, for max matchings M and M', a min M-stabilizer and a min M'-stabilizer can have a huge difference in size!

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

- From previous discussion, the difficulty lies in understanding which matching "survives" in $G \setminus F$!

- Take an arbitrary max matching. How bad can this choice be?

- Unfortunately, for max matchings M and M', a min M-stabilizer and a min M'-stabilizer can have a huge difference in size!

- A graph G is called ω-sparse if $\forall S \subseteq V$, $|E(S)| \leq \omega |S|$.

When M is NOT given

Thm: The stabilizer problem is NP-hard, and no $(2-\varepsilon)$-approximation exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture.

- What about approximation?

- From previous discussion, the difficulty lies in understanding which matching "survives" in $G \setminus F$!

- Take an arbitrary max matching. How bad can this choice be?

- Unfortunately, for max matchings M and M', a min M-stabilizer and a min M'-stabilizer can have a huge difference in size!

- A graph G is called ω-sparse if $\forall S \subseteq V$, $|E(S)| \leq \omega |S|$.

Thm: There is a 4ω-approximation algorithm for finding a minimum stabilizer.

Approximation Algorithm

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Lemma: Let G be a graph with $\nu_f(G) > \nu(G)$.

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Lemma: Let G be a graph with $\nu_f(G) > \nu(G)$. We can find $L \subseteq E$ with $|L| \leq 4\omega$ s.t.

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Lemma: Let G be a graph with $\nu_f(G) > \nu(G)$. We can find $L \subseteq E$ with $|L| \leq 4\omega$ s.t.

- $G \setminus L$ has a matching of size $\nu(G)$
- $\nu_f(G \setminus L) \leq \nu_f(G) - \frac{1}{2}$.

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Lemma: Let G be a graph with $\nu_f(G) > \nu(G)$. We can find $L \subseteq E$ with $|L| \leq 4\omega$ s.t.

- $G \setminus L$ has a matching of size $\nu(G)$
- $\nu_f(G \setminus L) \leq \nu_f(G) - \frac{1}{2}$.

- In other words, we can find a small subset of edges to remove from G that

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Lemma: Let G be a graph with $\nu_f(G) > \nu(G)$. We can find $L \subseteq E$ with $|L| \leq 4\omega$ s.t.

  - $G \setminus L$ has a matching of size $\nu(G)$
  - $\nu_f(G \setminus L) \leq \nu_f(G) - \frac{1}{2}$.

- In other words, we can find a small subset of edges to remove from G that
  - does not decrease the value of a max matching

Approximation Algorithm

- Our algorithm relies on the following Lemma.

Lemma: Let G be a graph with $\nu_f(G) > \nu(G)$. We can find $L \subseteq E$ with $|L| \leq 4\omega$ s.t.

- $G \setminus L$ has a matching of size $\nu(G)$
- $\nu_f(G \setminus L) \leq \nu_f(G) - \frac{1}{2}$.

- In other words, we can find a small subset of edges to remove from G that
  - does not decrease the value of a max matching
  - but reduces the minimum size of a fractional vertex cover.

Proof of the Lemma

Proof of the Lemma

Thm [Balas '81, Uhry '75]: One can find a half integral fractional matching x^* s.t.

(i) Edges $e : x^*_e = \frac{1}{2}$ form odd cycles $C_1, \dots, C_q$ with $q = 2|\nu_f(G) - \nu(G)|$

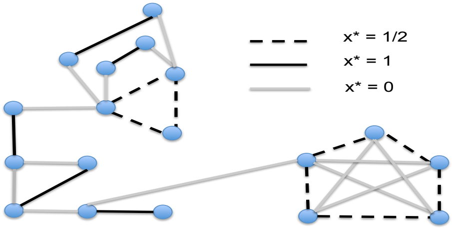
(ii) Let $\bar{M} := \{e \in E : x^*_e = 1\}$ and M_i be a maximum matching in C_i. Then $M' = \bar{M} \cup M_1 \cup \dots, \cup M_q$ is a maximum matching in G

Proof of the Lemma

Thm [Balas '81, Uhry '75]: One can find a half integral fractional matching x^* s.t.

(i) Edges $e : x_e^* = \frac{1}{2}$ form odd cycles $C_1, \ldots, C_q$ with $q = 2|\nu_f(G) - \nu(G)|$

(ii) Let $\bar{M} := \{e \in E : x_e^* = 1\}$ and M_i be a maximum matching in C_i. Then $M' = \bar{M} \cup M_1 \cup \ldots, \cup M_q$ is a maximum matching in G

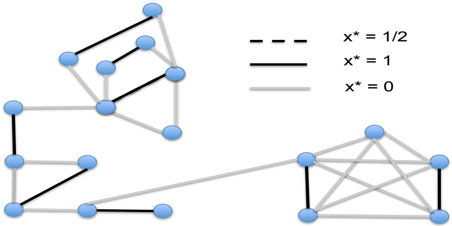


Proof of the Lemma

Thm [Balas '81, Uhry '75]: One can find a half integral fractional matching x^* s.t.

(i) Edges $e : x^*_e = \frac{1}{2}$ form odd cycles $C_1, \ldots, C_q$ with $q = 2|\nu_f(G) - \nu(G)|$

(ii) Let $\bar{M} := \{e \in E : x^*_e = 1\}$ and M_i be a maximum matching in C_i. Then $M' = \bar{M} \cup M_1 \cup \ldots, \cup M_q$ is a maximum matching in G

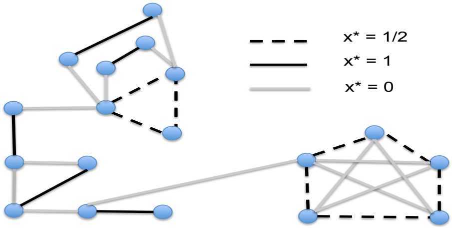


Proof of the Lemma

Thm [Balas '81, Uhry '75]: One can find a half integral fractional matching x^* s.t.

(i) Edges $e : x^*_e = \frac{1}{2}$ form odd cycles $C_1, \ldots, C_q$ with $q = 2|\nu_f(G) - \nu(G)|$

(ii) Let $\bar{M} := \{e \in E : x^*_e = 1\}$ and M_i be a maximum matching in C_i. Then $M' = \bar{M} \cup M_1 \cup \ldots, \cup M_q$ is a maximum matching in G

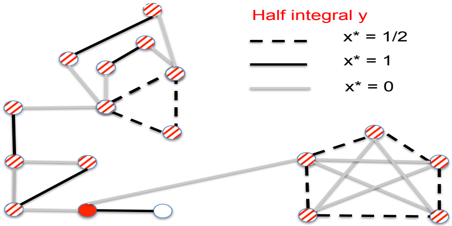


Proof of the Lemma

Thm [Balas '81, Uhry '75]: One can find a half integral fractional matching x^* s.t.

(i) Edges $e : x^*_e = \frac{1}{2}$ form odd cycles $C_1, \ldots, C_q$ with $q = 2|\nu_f(G) - \nu(G)|$

(ii) Let $\bar{M} := \{e \in E : x^*_e = 1\}$ and M_i be a maximum matching in C_i. Then $M' = \bar{M} \cup M_1 \cup \ldots, \cup M_q$ is a maximum matching in G



Proof of the Lemma

Proof of the Lemma

• Relying on complementary slackness, we find a half-integral y and show that there is a node u satisfying

Proof of the Lemma

• Relying on complementary slackness, we find a half-integral y and show that there is a node u satisfying

(a) $y_u = \frac{1}{2}$

(b) $L := \{\{u, w\} \in E : y_w = \frac{1}{2}\}$ satisfies $|L| \leq 4\omega$

(c) $\nu(G \setminus L) = \nu(G)$

Proof of the Lemma

• Relying on complementary slackness, we find a half-integral y and show that there is a node u satisfying

(a) $y_u = \frac{1}{2}$

(b) $L := \{\{u, w\} \in E : y_w = \frac{1}{2}\}$ satisfies $|L| \leq 4\omega$

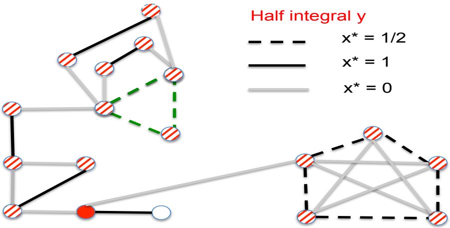
(c) $\nu(G \setminus L) = \nu(G)$

• Then, we just remove L and set $y_u := 0$!

Proof of the Lemma

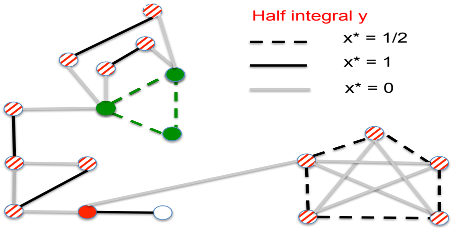
Proof of the Lemma

- We identify a set of nodes H with y-value equals $\frac{1}{2}$.



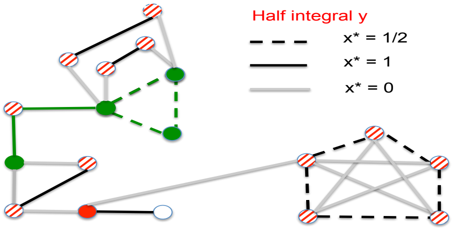
Proof of the Lemma

- We identify a set of nodes H with y-value equals $\frac{1}{2}$. Mark inessential nodes.



Proof of the Lemma

- We identify a set of nodes H with y-value equals $\frac{1}{2}$. Mark inessential nodes.



Approximation Algorithm

Thm: There is a 4ω-approximation algorithm for finding a minimum stabilizer.

Approximation Algorithm

Thm: There is a 4ω-approximation algorithm for finding a minimum stabilizer.

- For $i = 1, \ldots, 2(\nu_f(G) - \nu(G))$ do:
  (i) Apply previous Lemma.
  (ii) If G has become stable, STOP.

Approximation Algorithm

Thm: There is a 4ω-approximation algorithm for finding a minimum stabilizer.

- For $i = 1, \ldots, 2(\nu_f(G) - \nu(G))$ do:
  (i) Apply previous Lemma.
  (ii) If G has become stable, STOP.

- At the end of the Algorithm, we have a stable graph.

Approximation Algorithm

Thm: There is a 4ω-approximation algorithm for finding a minimum stabilizer.

- For $i = 1, \ldots, 2(\nu_f(G) - \nu(G))$ do:
  (i) Apply previous Lemma.
  (ii) If G has become stable, STOP.
- At the end of the Algorithm, we have a stable graph.
- We remove at most $4\omega \cdot 2(\nu_f(G) - \nu(G))$

Approximation Algorithm

Thm: There is a 4ω-approximation algorithm for finding a minimum stabilizer.

- For $i = 1, \ldots, 2(\nu_f(G) - \nu(G))$ do:
  - (i) Apply previous Lemma.
  - (ii) If G has become stable, STOP.

- At the end of the Algorithm, we have a stable graph.

- We remove at most $4\omega \cdot 2(\nu_f(G) - \nu(G))$

- It remains to observe that $2(\nu_f(G) - \nu(G))$ is a lower bound on the size of a min stabilizer!

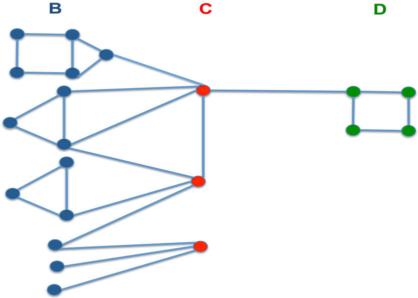
A lower bound

A lower bound

- Consider the Edmonds-Gallai decomposition [B,C,D] of G

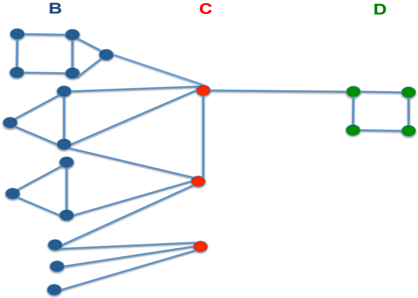
A lower bound

- Consider the Edmonds-Gallai decomposition [B,C,D] of G



A lower bound

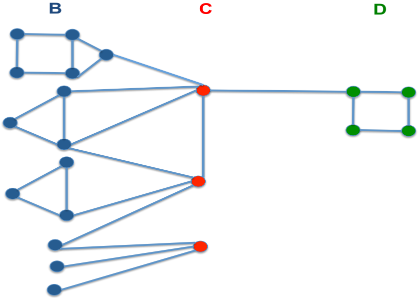
- Consider the Edmonds-Gallai decomposition [B,C,D] of G



- Compute a max matching M in G that exposes the fewest inessential singletons in B.

A lower bound

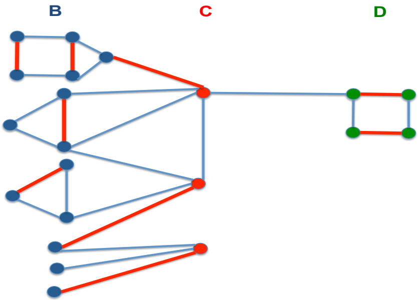
• Consider the Edmonds-Gallai decomposition [B,C,D] of G



• Compute a max matching M in G that exposes the fewest inessential singletons in B.

A lower bound

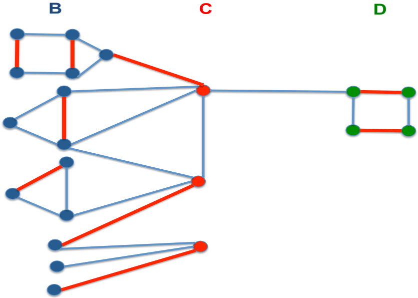
- Consider the Edmonds-Gallai decomposition [B,C,D] of G



- Compute a max matching M in G that exposes the fewest inessential singletons in B.

A lower bound

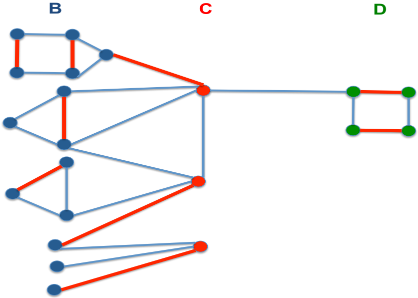
- Consider the Edmonds-Gallai decomposition [B,C,D] of G



- Compute a max matching M in G that exposes the fewest inessential singletons in B. $k := \#$ of exposed nodes in non-singleton components of B.

A lower bound

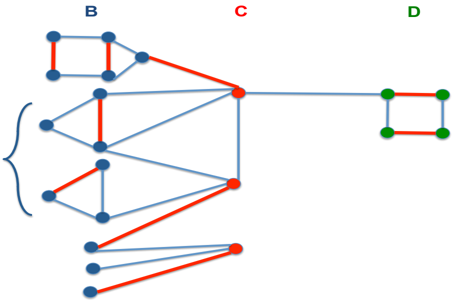
- Consider the Edmonds-Gallai decomposition [B,C,D] of G



- Compute a max matching M in G that exposes the fewest inessential singletons in B. k := # of exposed nodes in non-singleton components of B.

A lower bound

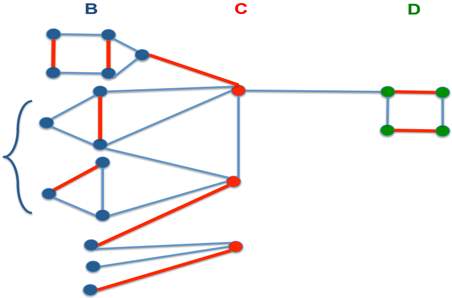
- Consider the Edmonds-Gallai decomposition [B,C,D] of G



- Compute a max matching M in G that exposes the fewest inessential singletons in B. $k := \#$ of exposed nodes in non-singleton components of B.

A lower bound

- Consider the Edmonds-Gallai decomposition [B,C,D] of G



- Compute a max matching M in G that exposes the fewest inessential singletons in B. $k := \#$ of exposed nodes in non-singleton components of B.

Thm [Pulleyblank '87]: $k = 2(\nu_f(G) - \nu(G))$

A lower bound

- Let F^* be a minimum stabilizer.

A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

- Let $H_1, \ldots, H_k$ be the k components in B defined by M.

A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

- Let $H_1, \ldots, H_k$ be the k components in B defined by M. Let M^* be max matching in $G \setminus F^*$.

A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

- Let $H_1, \ldots, H_k$ be the k components in B defined by M. Let M^* be max matching in $G \setminus F^*$.

Key observation: $\forall i$, there is at least one $v_i \in H_i$ that is essential in $G \setminus F^*$

A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

- Let $H_1, \ldots, H_k$ be the k components in B defined by M. Let M^* be max matching in $G \setminus F^*$.

Key observation: $\forall i$, there is at least one $v_i \in H_i$ that is essential in $G \setminus F^*$
...and w.l.o.g. we can assume each v_i is M-exposed!

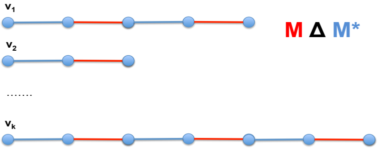
A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

- Let $H_1, \ldots, H_k$ be the k components in B defined by M. Let M^* be max matching in $G \setminus F^*$.

Key observation: $\forall i$, there is at least one $v_i \in H_i$ that is essential in $G \setminus F^*$ ...and w.l.o.g. we can assume each v_i is M-exposed!



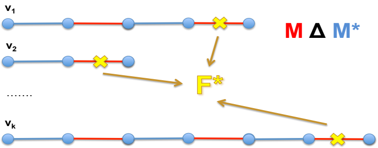
A lower bound

- Let F^* be a minimum stabilizer.

Claim: $k \leq |F^*|$

- Let $H_1, \ldots, H_k$ be the k components in B defined by M. Let M^* be max matching in $G \setminus F^*$.

Key observation: $\forall i$, there is at least one $v_i \in H_i$ that is essential in $G \setminus F^*$ ...and w.l.o.g. we can assume each v_i is M-exposed!



Final Remarks

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?
- For d-regular graphs, our algorithm yields a solution of value $k \cdot d$.

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?

- For d-regular graphs, our algorithm yields a solution of value $k \cdot d$. We get a 2-approximation algorithm by straightening the lower bound to $k\frac{d}{2}$.

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?

- For d-regular graphs, our algorithm yields a solution of value $k \cdot d$. We get a 2-approximation algorithm by straightening the lower bound to $k\frac{d}{2}$.

- ... what about factor critical graphs?

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?

- For d-regular graphs, our algorithm yields a solution of value $k \cdot d$. We get a 2-approximation algorithm by straightening the lower bound to $k\frac{d}{2}$.

- ... what about factor critical graphs?

- Good formulations?

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?

- For d-regular graphs, our algorithm yields a solution of value $k \cdot d$. We get a 2-approximation algorithm by straightening the lower bound to $k\frac{d}{2}$.

- ... what about factor critical graphs?

- Good formulations?

- Generalized settings?

Final Remarks

- **Main open question:** Is a $O(1)$-approximation possible?

- For d-regular graphs, our algorithm yields a solution of value $k \cdot d$. We get a 2-approximation algorithm by straightening the lower bound to $k\frac{d}{2}$.

- ... what about factor critical graphs?

- Good formulations?

- Generalized settings?

Thank you!