An Improved Model for Calculation of Debt Specific Risk VaR with Tail Fitting

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December 3, 2013

Disclaimer

OUTLINE

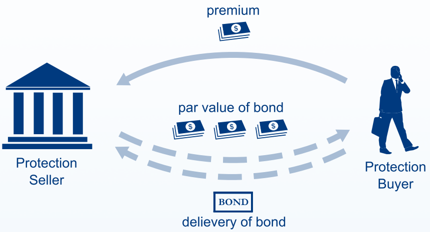
Debt Portfolio

A debt portfolio on the trading book is a portfolio consisting of the following instruments:

- Bond:
  - Corporate bonds
  - Agency bonds
  - Supanational bonds
  - Provincial/municipal bonds
  - Banker's acceptance
  - Non-domestic sovereign issues
  - etc.
- Single-name credit default swap (CDS)

Credit default swap

- A CDS is an instrument which provides a protection against the risk of a default on a bond issued by a reference entity
  - The protection buyer periodically pays premium (CDS spread) of X bps in annual basis until the maturity or the occurrance of default of the reference entity, whichever is earlier;
  - If the reference entity defaults, the price of bonds issued by the reference entity collapse. The CDS contract provides the issurance for the protection buyer:
    - The protection seller pays the par value of the bond;
    - The protection buyer delivers the bond.



- The use of CDS
  - Hedge: protect againt the default of the reference entity of a bond;
  - Speculation: bet on the health of the reference entity, may not hold any bond issued by the reference entity;
  - Arbitrage: capital structure arbitrage, etc.

Risk in Debt Portfolio

- Risk embedded in positions (bond or CDS) of debt portfolio includes:
  - Market risk: change of PnL due to systematic risk factors that affect the overall performance of the financial markets, such as interest rates;
  - Specific risk: change of PnL to idiosyncratic risk factors which exclude credit events;
  - Migration/default risk: change of PnL due to the change of the credit rating of a bond or default on a bond;
  - Other risk: liquidity risk, counterparty credit risk (OTC trades), etc.
- Time horizon for different risks
  - Market risk and specific risk cover price volatility that would normally occrur over a short period (e.g. 10 days);
  - Migration/defulat risk captures migration and default risk over a longer period (e.g. 1 year).
- Since Basel 2.5, banks are required to
  - Develop an Incremental Risk Charge (IRC) model to calculate capitals reserved for migration/default risk;
  - Modify the existing risk model to account for both market risk and specific risk.

Total VaR

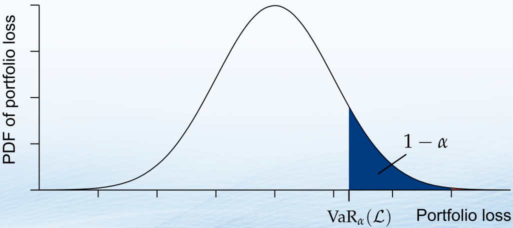
- A term "total risk" is proposed to cover market risk and specific risk:

$$\text{Total risk} = \text{Market risk} + \text{Specific risk}.$$

- The value-at-risk (VaR) measure is used to measure total risk;
- VaR of a portfolio is defined by

$$\text{VaR}_\alpha(\mathcal{L}) = \inf\{l \in \mathbb{R} : \mathbb{P}[\mathcal{L} \leq l] \geq \alpha\},$$

  - $\mathcal{L}$ is the portfolio loss;
  - α is the predetermined confidence level, for total risk, $\alpha = 99\%$;
  - VaR can be interpreted as "We are α certain that we will not lose more than $\text{VaR}_\alpha(\mathcal{L})$ dollars in the considered time horizon."
  - Mathematically, VaR of $\mathcal{L}$ with confidence level α is the α-quantile of $\mathcal{L}$:



Outline

Decomposition of Spread

- Denote y and r as a bond's yield and the corresponding benchmark government bond yield respectively, and the yield spread is $y - r$;
- Let s be the spread of a CDS;
- The day-over-day change of the spread,

$$z = \begin{cases} \Delta(y - r), & \text{for bond} \\ \Delta s, & \text{for CDS} \end{cases}$$

  can be decomposed by

$$z = \beta\tilde{z} + \epsilon;$$

  - $\tilde{z}$: the average spread change of bonds (or CDSs) which are within the same currency/sector/rating category and have similar tenor;
  - β: the sensitivity of the spread change of the bond (or the CDS) to the average spread change;
  - ϵ: the idiosyncratic spread change (residual), of the bond (or the CDS);
  - $\tilde{z}$ and ϵ are assumed to be independent;
- The risk due to $\beta\tilde{z}$ is captured in market risk model;
- The specific risk model examine the risk due to residuals ϵ.

Position Loss

Let $\mathcal{P}$ be a position's daily PnL, then a position's daily loss due to the idiosyncratic risk factor can be approximated by

- Delta approximation (first-order)

$$\mathcal{L} \approx - \left.\frac{\partial \mathcal{P}}{\partial \epsilon}\right|_{\epsilon=0} \cdot \epsilon = -\delta\sigma\bar{\epsilon},$$

  where $\delta = \left.\frac{\partial \mathcal{P}}{\partial \epsilon}\right|_{\epsilon=0}$, σ is the standard deviation of ϵ, and $\bar{\epsilon}$ is the normalized residual;

  - Closed-form distribution for portfolio loss under certain assumption on the distribution of residuals;
  - Linear loss approximation, risk beyond the first order is ignored;

- Delta-Gamma approximation (second-order)

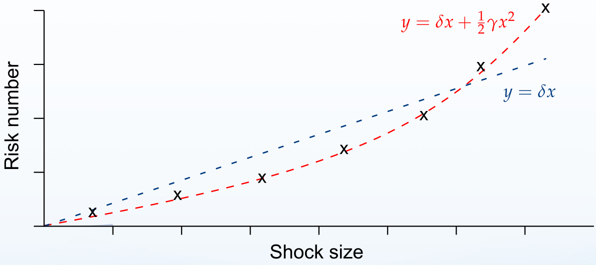
$$\mathcal{L} \approx - \left(\left.\frac{\partial \mathcal{P}}{\partial \epsilon}\right|_{\epsilon=0} \cdot \epsilon + \frac{1}{2} \left.\frac{\partial^2 \mathcal{P}}{\partial \epsilon^2}\right|_{\epsilon=0} \cdot \epsilon^2 \right) = -\delta\sigma\bar{\epsilon} - \frac{1}{2}\gamma\sigma^2\bar{\epsilon}^2,$$

  where $\gamma = \left.\frac{\partial^2 \mathcal{P}}{\partial \epsilon^2}\right|_{\epsilon=0}$;

  - No closed-form distribution for portfolio loss, Monte Carlo simulation is needed;
  - Second order approximation, more accruate.

Estimation of Delta and Gamma

- Most risk engines are capable to provide sensitivity of positions to shocks applied to credit spreads;
  - Risk number, $csPV_{m_j}, j = 1, \ldots, J$: the position PnL if an absolute shock of m_j bp is applied to the credit spread;
- δ and γ can be estimated by linear squares regression:



Estimation of Delta and Gamma (cont'd)

- δ in Delta approximation can be estimated by:

$$\min_{\delta} \sum_{j=1}^{J} \left(csPV_{m_j} - \delta m_j \right)^2$$

  - Solution is

$$\delta = (X'X)^{-1}X'Y = \frac{\sum_{j=1}^{J} m_j csPV_{m_j}}{\sum_{j=1}^{J} m_j^2},$$

  where $X = [m_1, \cdots, m_J]'$ and $Y = \left[csPV_{m_1}, \cdots, csPV_{m_J}\right]'$;

- δ and γ in Delta-Gamma approximation can be estimated by:

$$\min_{\delta,\gamma} \sum_{j=1}^{J} \left(csPV_{m_j} - \left(\delta m_j + \frac{1}{2}\gamma m_j^2 \right) \right)^2$$

  - Solution is

$$\begin{bmatrix} \delta \\ \gamma \end{bmatrix} = (X'X)^{-1}X'Y,$$

  where $X = \begin{bmatrix} m_1 & \frac{1}{2} m_1 \\ \vdots & \vdots \\ m_J & \frac{1}{2} m_J \end{bmatrix}$, and $Y = \begin{bmatrix} csPV_{m_1} \\ \vdots \\ csPV_{m_J} \end{bmatrix}$.

Position Mapping in Debt Portfolio

- Positions in a debt portfolio have
  - Different issuer/reference entity;
  - Different tenor/maturity;
- Positions with different issuer/reference entitys have different marginal residual distribution;
- Positions with same issuer/reference entity but different tenor/maturity may have a very different marginal residual distribution as well;
- However, it is not practical to model every position's marginal residual distrubution:
  - Missing data;
  - Too computatinally intense;

Position Mapping in Debt Portfolio (Cont'd)

- Tenors/maturities can be mapped to "proxy tenors"

Position tenor	Proxy tenor
[0yr, 1yr)	1yr
[1yr, 3yr)	2yr
[4yr, 9yr)	5yr
[9yr, 15yr)	10yr
[15yr, $+\infty$)	20yr

- Positions are grouped into categories $\mathbf{K}_{n,m,j}$, $n = 1, \ldots, N$, $m = 1, \ldots, 5$ and $j = 1, 2$, where

$$\mathbf{K}_{n,m,1} = \{k \mid \text{position } k \text{ is a bond with the } m\text{th proxy tenor and the } n\text{th issuer}\},$$
$$\mathbf{K}_{n,m,2} = \{k \mid \text{position } k \text{ is a CDS with the } m\text{th proxy tenor and the } n\text{th reference entity}\}.$$

- Positions within the same subset share a common residual.

PORTFOLIO SR LOSS

The h-day portfolio loss is computed by

- Delta approximation:

$$\mathcal{L} \approx -\sum_{n=1}^{N}\sum_{m=1}^{5}\sum_{j=1}^{2}\left(\sqrt{h}\tilde{\delta}_{n,m,j}\right)\bar{\epsilon}_{n,m,j}$$

  where $\tilde{\delta}_{n,m,j} = \sum_{k\in\mathbf{K}_{n,m,j}} \delta_k\sigma_k$;

- Delta-Gamma approximation

$$\mathcal{L} \approx -\sum_{n=1}^{N}\sum_{m=1}^{5}\sum_{j=1}^{2}\left(\left(\sqrt{h}\tilde{\delta}_{n,m,j}\right)\bar{\epsilon}_{n,m,j} + \frac{1}{2}\left(h\tilde{\gamma}_{n,m,j}\right)\bar{\epsilon}^2_{n,m,j}\right)$$

  where $\tilde{\gamma}_{n,m,j} = \sum_{k\in\mathbf{K}_{n,m,j}} \gamma_k\sigma_k^2$.

Portfolio Total VaR

- The total VaR with the confidence level α is defined by

$$\mathrm{VaR}_\alpha\left(\mathcal{L}_{TR}\right) := \inf\left\{q \in \mathbb{R} : \mathbb{P}\left[\mathcal{L}_{TR} \leq q\right] \geq \alpha\right\};$$

- The total portfolio loss can be approximated by the summation of the loss calculated in the market risk model and the SR loss:

$$\mathcal{L}_{TR} \approx \mathcal{L}_{MR} + \mathcal{L}_{SR};$$

- A scenario-based market risk model usually generates I scenario losses:

$$\mathcal{L}_{MR}^{(1)}, \ldots, \mathcal{L}_{MR}^{(I)};$$

- Assuming that shocks on residuals are independent with shocks on other market risk factors, $\mathcal{L}_{MR}$ are independent with $\mathcal{L}_{SR}$. Hence,

$$\begin{aligned}\mathbb{P}\left[\mathcal{L}_{TR} \leq q\right] &= \mathbb{P}\left[\mathcal{L}_{MR} + \mathcal{L}_{SR} \leq q\right] \\ &= \mathbb{E}\left[\mathbb{P}\left[\mathcal{L}_{MR} + \mathcal{L}_{SR} \leq q \middle| \mathcal{L}_{MR}\right]\right] \\ &\approx \frac{1}{I}\sum_{i=1}^{I}\mathbb{P}\left[\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}\right];\end{aligned}$$

- Rest of the problem: model the distribution of residuals to calculate

$$\mathbb{P}\left[\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}\right].$$

Outline

Existing Distributions

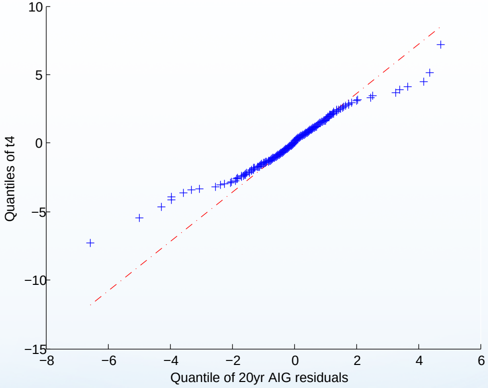
- In practice, normalized residuals $\bar{\epsilon}$ are usually modeled by the multi-variate student's t distribution with DoF ν;
- If the Delta approximation is applied:
  - The portfolio SR loss, $\mathcal{L}_{SR}$, is a linear combination of random varibles subject to multi-variate student's t distribution;
  - Consequently, $\sqrt{\frac{\nu}{\nu-2}}\frac{\mathcal{L}_{SR}}{\sqrt{h}\sigma_P}$ is a uni-variate student t distribution with the same degree of freedom ν;
    - $\sigma_P = \sqrt{h}\sqrt{\tilde{\delta}^T \boldsymbol{\rho} \tilde{\delta}}$ is the h-day portfolio SR PnL volatility;
    - $\boldsymbol{\rho}$ is the correlation matrix of $\bar{\epsilon}$;
  - The probability $\mathbb{P}\left(\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}\right)$ can be computed analytically:

$$\mathbb{P}\left(\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}\right) = t_\nu\left(\frac{\sqrt{\frac{\nu}{\nu-2}}\cdot\left(q - \mathcal{L}_{MR}^{(i)}\right)}{\sqrt{h}\sigma_P}\right);$$

- No closed-form solution for the Delta-Gamma approximation.

Existing Distributions (Cont'd)

- The marginal student's t distribution may not be close to the distribution of some bond/CDS residuals:



Empirical Distribution

- Given historical data of normalized residuals, $\bar{\epsilon}_{n,m,j}^{(1)}, \ldots, \bar{\epsilon}_{n,m,j}^{(U)}$, we can compute the empirical CDF for $\bar{\epsilon}_{n,m,j}$:

$$\tilde{F}_{n,m,j}(x) = \frac{1}{U} \sum_{u=1}^{U} \mathbb{I}_{\left\{\bar{\epsilon}_{n,m,j}^{(u)} \leq x\right\}},$$

where $\mathbb{I}_A$ is an indicator variable

$$\mathbb{I}_A = \begin{cases} 1, & \text{if } A \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

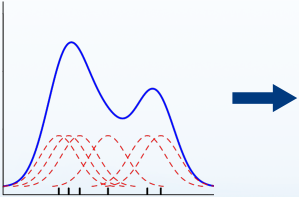
- We could use the empirical distribution implied by $\bar{\epsilon}_{n,m,j}^{(1)}, \ldots, \bar{\epsilon}_{n,m,j}^{(U)}$ as the marginal distribution of $\bar{\epsilon}_{n,m,j}$, BUT
  - The empirical distribution is not continuous, which is not desirable from the aspect of sampling;
  - Sampling from an empirical distribution implied by U unique observations generates at most U unique samples.

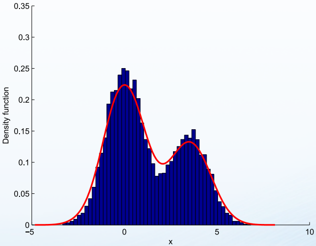
Kernel Density Estimation

- Given a series of observations $x^{(1)}, \ldots, x^{(U)}$, the kernel density estimator can be used to estimate the unknown density:

$$\hat{f}_h(x) = \frac{1}{Uh} \sum_{u=1}^{U} K\left(\frac{x - x^{(u)}}{h}\right);$$

  - $K(\cdot)$ is the kernel function, which determines the shape of the density;
  - h is the bandwidth or smoothing constant, which determines the smoothness of the density.





Normal Kernal Distribution (NK)

- Normal kernel:

$$K(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2};$$

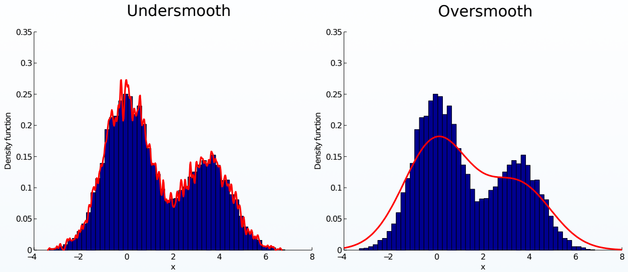
- Normal kernel CDF estimator for the marginal distribution of normalized residuals:

$$\hat{F}_{n,m,j}(x) = \frac{1}{Uh} \sum_{u=1}^{U} \Phi\left(\frac{x - \bar{\epsilon}_{n,m,j}^{(i)}}{h}\right),$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution;

Bandwidth Selection

- The bandwidth h determines the quality of the estimation
  - Larger bandwidth: less variance but more bias;
  - Smaller bandwidth: less bias but more variance;
- The bandwidth h determines the smoothness of the density:
  - Larger bandwidth: smoother density estimation;



- A rule of thumb for normal kernel: "Silverman's rule of thumb"

$$h = \left(\frac{4\hat{\sigma}^5}{3U}\right)^{1/5} \approx 1.06\hat{\sigma}U^{-1/5},$$

where $\hat{\sigma}$ is the sample standard deviation.

Pareto Distribution

- The normal kernel with the bandwidth by Silverman's rule of thumb usually generates
  - Well-suited estimates for densities in the middle portion of the distribution;
  - Under-smoothed, high variance tails;
- To better estimate the tails of the distribution, the generalized Pareto (GP) distribution can be used to model the distribution of exceedances of residuals over pre-determined thresholds;
- The density of the GP distribution with shape parameter ξ, scale parameter σ and location parameter μ, is

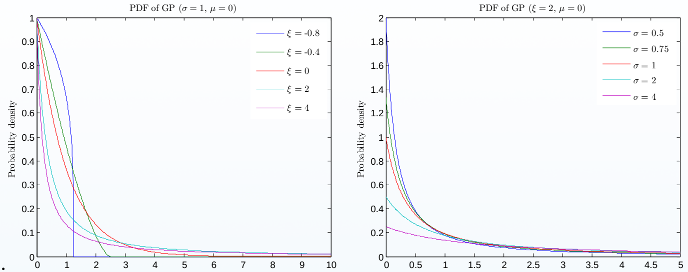
$$g(x|\xi,\sigma,\mu) = \begin{cases} \frac{1}{\sigma}\left(1+\frac{\xi(x-\mu)}{\sigma}\right)^{-(1+1/\xi)} & \text{for } x \geq \mu \text{ when } \xi > 0, \text{ or for } \mu \leq x \leq \mu - \sigma/\xi \text{ when } \xi < 0, \\ \frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}} & \text{for } x \geq \mu \text{ when } \xi = 0, \\ 0 & \text{otherwise,} \end{cases}$$

  - ξ: shape parameter;
  - σ: scale parameter;
  - μ: location parameter;
- The CDF of GP distribution is

$$G(x|\xi,\sigma,\mu) == \begin{cases} 1-\left(1+\frac{\xi(x-\mu)}{\sigma}\right)^{-1/\xi} & \text{for } x \geq \mu \text{ when } \xi > 0, \text{ or for } \mu \leq x \leq \mu - \sigma/\xi \text{ when } \xi < 0, \\ 1-e^{-\frac{x-\mu}{\sigma}} & \text{for } x \geq \mu \text{ when } \xi = 0, \\ 0 & \text{for } x < \mu, \\ 1 & \text{otherwise.} \end{cases}$$

Pareto Distribution (Cont'd)

- Capable to fit a wide variety of fat-tailed data



PARETO TAILS

- Upper tail:
  - Select an upper tail threshold $\grave{\alpha}$, e.g. $\grave{\alpha} = 90\%$;
  - Calculate the $\grave{\alpha}$ quantile of the normal kernel distribution, $\grave{Q}_{n,m,j}$;
  - Calculate upper exceedances, $\grave{\iota}^{(i)}_{n,m,j}$, by :

$$\grave{\iota}^{(i)}_{n,m,j} = \bar{\epsilon}^{(i)}_{n,m,j} - \grave{Q}_{n,m,j}, \text{ for } i \in \grave{\mathbf{S}}_{n,m,j} = \left\{ i \,\middle|\, \bar{\epsilon}^{(i)}_{n,m,j} > \grave{Q}_{n,m,j} \right\};$$

  - Choose a proper GP distribution, $GP\left(\grave{\xi}_{n,m,j}, \grave{\sigma}_{n,m,j}, 0\right)$, to fit $\grave{\iota}^{(i)}_{n,m,j}$ by the maximum likelihood estimation (MLE):

$$\left(\grave{\xi}_{n,m,j}, \grave{\sigma}_{n,m,j}\right) := \arg\max_{\xi,\sigma} \sum_{i \in \grave{\mathbf{S}}_{n,m,j}} \ln g\left(\grave{\iota}^{(i)}_{n,m,j} \,\middle|\, \xi, \sigma, 0 \right);$$

  - The optimization problem can be solved by Nelder–Mead method;
- Lower tail:
  - Select a lower tail threshold $\acute{\alpha}$, e.g. $\acute{\alpha} = 10\%$;
  - Calculate the $\acute{\alpha}$ quantile of the normal kernel distribution, $\acute{Q}_{n,m,j}$;
  - Calculate lower exceedances, $\acute{\iota}^{(i)}_{n,m,j}$, by:

$$\acute{\iota}^{(i)}_{n,m,j} = \acute{Q}_{n,m,j} - \bar{\epsilon}^{(i)}_{n,m,j}, \text{ for } i \in \acute{\mathbf{S}}_{n,m,j} = \left\{ i \,\middle|\, \bar{\epsilon}^{(i)}_{n,m,j} < \acute{Q}_{n,m,j} \right\};$$

  - Choose a proper GP distribution, $GP\left(\acute{\xi}_{n,m,j}, \acute{\sigma}_{n,m,j}, 0\right)$, to fit $\acute{\iota}^{(i)}_{n,m,j}$ by MLE:

$$\left(\acute{\xi}_{n,m,j}, \acute{\sigma}_{n,m,j}\right) := \arg\max_{\xi,\sigma} \sum_{i \in \acute{\mathbf{S}}_{n,m,j}} \ln g\left(\acute{\iota}^{(i)}_{n,m,j} \,\middle|\, \xi, \sigma, 0 \right).$$

Normal Kernel Distribution with Pareto Tails (NKPT)

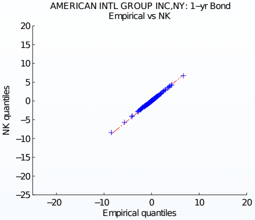
- Combining the normal kernel distribution and the Pareto tails enable us to model the distribution of residuals by the following semi-parametric model:

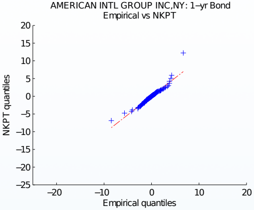
$$\bar{\epsilon}_{n,m,j} = \left(\acute{Q}_{n,m,j} - \mathcal{Y}\right) \cdot \mathbb{I}_{\left\{\mathcal{X} \in \left(-\infty, \acute{Q}_{n,m,j}\right)\right\}} + \mathcal{X} \cdot \mathbb{I}_{\left\{\mathcal{X} \in \left[\acute{Q}_{n,m,j}, \grave{Q}_{n,m,j}\right]\right\}} + \left(\grave{Q}_{n,m,j} + \mathcal{Z}\right) \cdot \mathbb{I}_{\left\{\mathcal{X} \in \left(\grave{Q}_{n,m,j}, +\infty\right)\right\}},$$

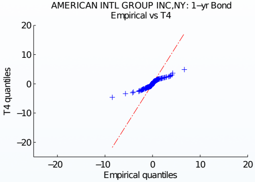
  - $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are mutually independent;
  - $\mathcal{X}$ is subject to the normal kernel distribution with CDF $\hat{F}_{n,m,j}(x)$;
  - $\mathcal{Y}$ follows GP distribution with CDF $G\left(x \,\middle|\, \acute{\xi}_{n,m,j}, \acute{\sigma}_{n,m,j}, 0\right)$;
  - $\mathcal{Z}$ follows GP distribution with CDF $G\left(x \,\middle|\, \grave{\xi}_{n,m,j}, \grave{\sigma}_{n,m,j}, 0\right)$.
- The CDF of the NKPT distribution is a piecewise function:

$$F_{n,m,j}(x) = \begin{cases} \acute{\alpha}\left(1 - G\left(\acute{Q}_{n,m,j} - x \,\middle|\, \acute{\xi}_{n,m,j}, \acute{\sigma}_{n,m,j}, 0\right)\right), & x \in \left(-\infty, \acute{Q}_{n,m,j}\right), \\ \hat{F}_{n,m,j}(x), & x \in \left[\acute{Q}_{n,m,j}, \grave{Q}_{n,m,j}\right], \\ \grave{\alpha} + (1 - \grave{\alpha})\, G\left(x - \grave{Q}_{n,m,j} \,\middle|\, \grave{\xi}_{n,m,j}, \grave{\sigma}_{n,m,j}, 0\right), & x \in \left(\grave{Q}_{n,m,j}, +\infty\right). \end{cases}$$

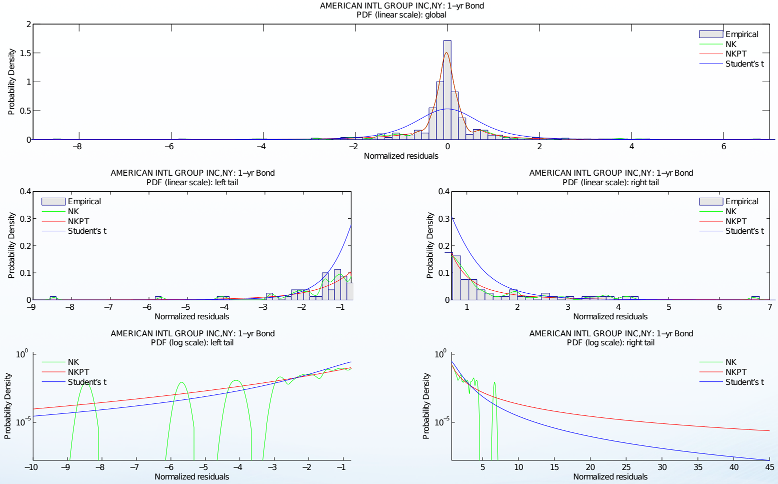
Comparison of Marginal Distributions







Comparison of Marginal Distributions (Cont'd)



Joint Distribution: Copula

- Copula is usually used to construct the joint distribution from marginal distributions;
- A copula is defined as a distribution on the unit cube $[0,1]^N$:

$$C(u_1, u_2, \ldots, u_N) = \mathbb{P}\left[\mathcal{U}_1 \le u_1, \mathcal{U}_2 \le u_2, \ldots, \mathcal{U}_N \le u_N\right];$$

  - E.g. given a correlation matrix $\boldsymbol{\rho}$, the student's t copula with 4 DoF can be written as

$$C^{\rho}_{t_4}(u_1, u_2, \ldots, u_N) = t^{\rho}_4\left(t_4^{-1}(u_1), t_4^{-1}(u_2), \ldots, t_4^{-1}(u_N)\right);$$

- Consider a random vector $[\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]'$ with continuous marginal distribution F_i:
  - Transform $\mathcal{X}_i$ to $\mathcal{Y}_i$ by

$$\mathcal{Y}_i = t_4^{-1}(F_i(\mathcal{X}_i)),$$

  - Assume $[\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N]'$ follows a multi-variate T4 distribution with the correlation matrix $\boldsymbol{\rho}$, then the joint distribution of $[\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]'$ can be written as

$$\mathbb{P}\left[\mathcal{X}_1 \le x_1, \mathcal{X}_2 \le x_2, \ldots, \mathcal{X}_N \le x_N\right] = C^{\rho}_{t_4}\left(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)\right);$$

  - The marginal distribution of $\mathcal{X}_i$ is preserved while defining a correlation structure of $[\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]'$ via the correlation structure of $[\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N]'$.

STUDENT'S T COPULA NKPT

- Normalized residuals, $\bar{\epsilon}_{n,m,j}$, are modeled by

$$\bar{\epsilon}_{n,m,j} = F^{-1}_{n,m,j}\left(t_4\left(\omega_{n,m,j}\right)\right)$$

  - The marginal distribution of $\bar{\epsilon}_{n,m,j}$ is the NKPT distribution with CDF $F_{n,m,j}(x)$;
  - A student's t copula with 4 DoF is used for the joint distribution:
    - The intermediate random vector $\boldsymbol{\omega}$ follows a multi-vairate T4 distribution with a correlation matrix $\boldsymbol{\varrho}$;
    - $\boldsymbol{\varrho}$ is the correlation matrix of normalized residuals $\bar{\epsilon}$.
- No analytical solution for the distribution of the portfolio SR loss;
- Instead, MC simulation is needed to calcalute $\mathbb{P}\left(\mathcal{L}_{SR} \leq q - \mathcal{L}^{(i)}_{MR}\right)$;

MC DSR WITH STUDENT'S T COPULA NKPT

- Sample $\boldsymbol{\omega}$ from the multi-vairate T4 distribution with the correlation matrix $\boldsymbol{\varrho}$: $\boldsymbol{\omega}^{(1)}, \ldots, \boldsymbol{\omega}^{(K)}$;
- Calculate $\bar{\boldsymbol{\epsilon}}^{(1)}, \ldots, \bar{\boldsymbol{\epsilon}}^{(K)}$ by $\bar{\epsilon}_{n,m,j}^{(k)} = F_{n,m,j}^{-1}\left(t_4\left(\omega_{n,m,j}^{(k)}\right)\right)$;
- Compute portfolio SR losses by Delta approximation

$$\mathcal{L}_{SR}^{(k)} = -\sum_{n=1}^{N}\sum_{m=1}^{5}\sum_{j=1}^{2}\left(\sqrt{h}\tilde{\delta}_{n,m,j}\right)\bar{\epsilon}_{n,m,j}^{(k)},$$

  or Delta-Gamma approximation

$$\mathcal{L}_{SR}^{(k)} = -\sum_{n=1}^{N}\sum_{m=1}^{5}\sum_{j=1}^{2}\left(\left(\sqrt{h}\tilde{\delta}_{n,m,j}\right)\bar{\epsilon}_{n,m,j}^{(k)} + \frac{1}{2}\left(h\tilde{\gamma}_{n,m,j}\right)\left(\bar{\epsilon}_{n,m,j}^{(k)}\right)^2\right);$$

- Approximate $\mathbb{P}\left(\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}\right)$ by

$$\mathbb{P}\left(\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}\right) \approx \frac{1}{K}\sum_{k=1}^{K}\mathbb{I}_{\left\{\mathcal{L}_{SR}^{(k)} \leq q - \mathcal{L}_{MR}^{(i)}\right\}}.$$

Testing Results: 99% TR VaR

Model	Delta	Delta-Gamma
T4	15,230,674.10	15,053,488.33
Copula NK	15,733,387.31	15,565,004.61
Copula NKPT	15,809,243.83	15,516,868.54

- Assumption of mutli-variate student's t distribution underestimates the risk (about half million for the testing portfolio);
- Compared with the Delta approximation, the Delta-Gamma approximation lowers the VaR number.