The Term Structure of Variance Swaps, Risk Premia and the Expectation Hypothesis

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1. Introduction

- Among volatility derivatives, Variance Swaps are simple contracts:
  - the fixed leg agrees at inception that it will pay a fixed amount at maturity T, the VS rate
  - in exchange to receiving a floating amount based on the realized variance of the underlying asset from 0 to T.
- One potential difficulty lies in the path-dependency introduced by the realized variance.

- But the payoff of a VS can be replicated by dynamic trading in the underlying asset and a static position in vanilla options
  - This insight, originally due to Neuberger (1994) and Dupire (1993), means that the path-dependency implicit in VS could be circumvented
  - It also made possible the analysis of the various hedging errors when attempting to replicate a given VS (see, e.g., Carr and Madan (1998), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Jiang and Oomen (2008), Carr and Wu (2009), Carr and Lee (2010).)
  - Because of the interest in replicating a given contract, VS rates have generally been studied at a single maturity.
  - Including a determination of the variance risk premia at a single short maturity: see Carr and Wu (2009), Todorov (2010), Bollerslev and Todorov (2011a) and Bollerslev and Todorov (2011b).

- But VS rates give rise naturally to a term structure, by varying the maturity at which the exchange of cashflows take place
  - The literature has proceded by analogy with the term structure of interest rates
  - Including determining the number of factors necessary to capture the variation of the curve (see Bühler (2006), Gatheral (2008), Amengual (2008) and Egloff et al. (2010).)

- We continue this line of research with two differences:
  - First, we do not proceed fully by analogy with the term structure of interest rates, i.e., by taking either the variances themselves or their latent factors as the primitives
    * Instead, we incorporate the fact that the variance in a VS is that of an underlying asset
    * This means that we can infer properties of the risk premia associated not just with the variances but also with the asset itself, which in the case of the S&P500 is the classical equity risk premium.
  - Second, we allow for the presence of jumps in asset returns.

- Various aspects of the VS term structure cannot be studied in a model-free manner, because the necessary data are either insufficient in quantity or simply unavailable.
  - We work with a parametric stochastic volatility model which is consistent with the salient empirical features of VS rates documented in the model-free analysis.
  - The model is estimated using maximum-likelihood, combining time series information on stock returns and cross sectional information on the term structure of VS rates.

- The model allows the risk premia to be estimated in a term structure framework
  - the variance risk premium
  - and the equity risk premium
- We study whether the Expectation Hypothesis (EH) holds in the term structure of VS.
- Finally, to assess the economic profitability of VS contracts, we develop a simple but robust trading strategy involving VS: as the ex-ante variance risk premium is found to be negative, the strategy takes a short position in the VS contract .

2. Variance Swap Rates

- Let S be a semimartingale modeling the stock (or index) price process with dynamics

$$dS_t/S_{t\_} = \mu_t\,dt + v_t^{1/2}\,d\tilde{W}_t^P + (\exp(J_t^P) - 1)\,dN_t^P - \nu_t^P\,dt$$

- The dynamics of the drift, variance, and jump component are left unspecified: model-free.

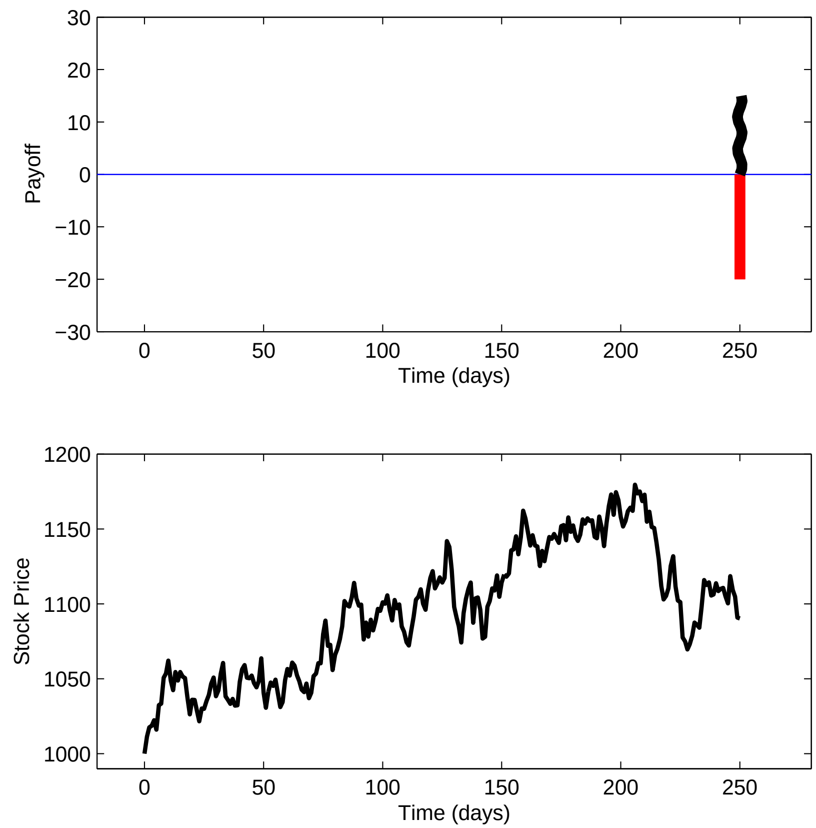
- The floating leg of the swap pays at $t + \tau$:

$$\mathrm{RV}_{t,t+\tau} = \frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2.$$

- Like any swap, no cashflow changes hands at inception of the contract at time t; the fixed leg of the variance swap agrees to pay an amount fixed at time t, defined as the variance swap rate, $\mathrm{VS}_{t,t+\tau}$.
  - Therefore, at maturity, $t+\tau$, the long position in a variance swap contract receives the difference between the realized variance between times t and $t+\tau$, $\mathrm{RV}_{t,t+\tau}$, and the variance swap rate, $\mathrm{VS}_{t,t+\tau}$, which was fixed at time t.
  - The difference is multiplied by a fixed notional amount to convert the payoff to dollar terms:

$$(\mathrm{RV}_{t,t+\tau} - \mathrm{VS}_{t,t+\tau}) \times \text{notional}.$$

VS Payoffs



Payoff
30
20
10
0
−10
−20
−30
0
50
100
150
200
250
Time (days)
Stock Price
1200
1150
1100
1050
1000
0
50
100
150
200
250
Time (days)

- The analysis of variance swap contracts is simplified when the realized variance is replaced by the quadratic variation of the log-price process.

- It is well known that when $\sup_{i=1,\ldots,n}(t_i - t_{i-1}) \to 0$

$$\frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 \longrightarrow \frac{1}{\tau}\int_t^{t+\tau} v_s\, ds + \frac{1}{\tau}\sum_{u=N_t}^{N_{t+\tau}} J_u^2 = \mathsf{QV}_{t,t+\tau}$$

- As usual, we assume absence of arbitrage, which implies the existence of an equivalent risk-neutral measure Q and:

$$\mathsf{VS}_{t,t+\tau} = E_t^Q[\mathsf{QV}_{t,t+\tau}]$$

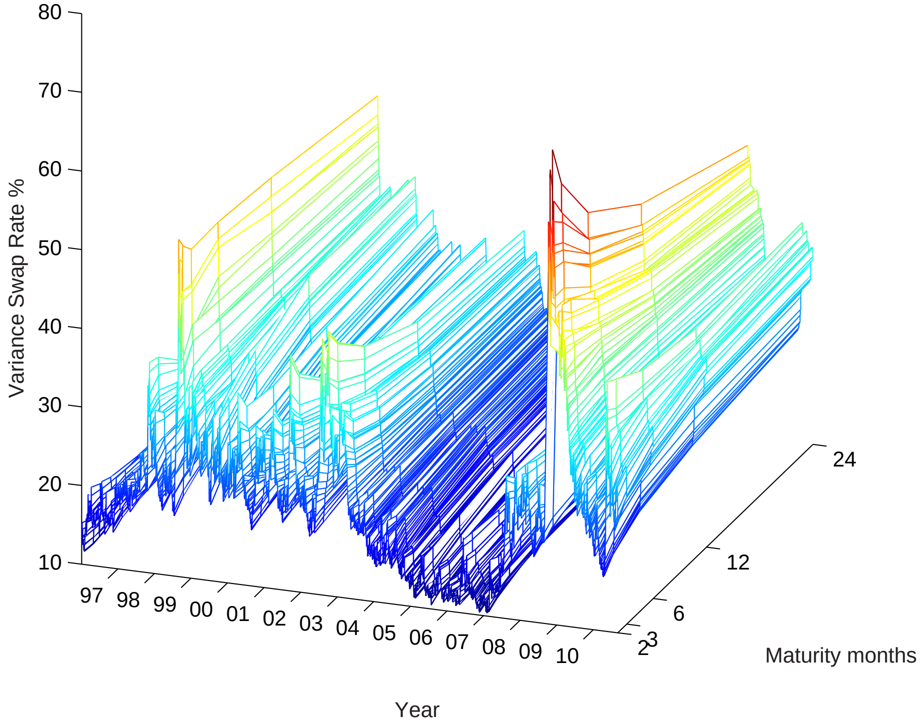
- The dependence in τ produces the term structure.

2.1. Preliminary Data Analysis

- Dataset: OTC quotes on VS on the S&P500 provided by a major broker-dealer in New York City.

- The data are daily closing quotes on variance swap rates with fixed time to maturities of 2, 3, 6, 12, and 24 months from January 4, 1996 to September 2, 2010, resulting in 3,624 observations for each maturity.

The Term Structure of VS Rates



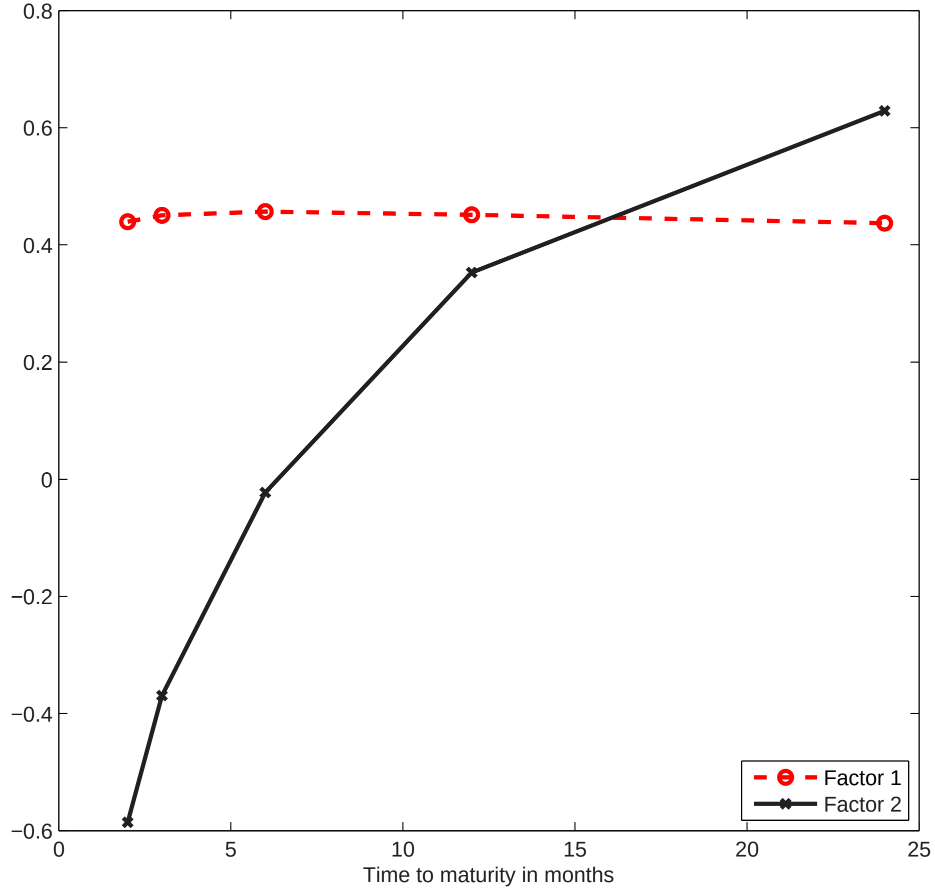
- VS rates are mean-reverting, volatile, and exhibitspikes and clustering.

- While most term structures are upward sloping (53% of the sample), they are often ∪-shape too (23% of the sample). The rest are downward sloping and ∩-shape term structures.

- In turbulent times, the short-end of the term structure (VS rates with 2 or 3 months to maturity) rises more than the long-end, producing downward sloping term structures.

- Principal Component Analysis
  - 1st explains 95.4% of the total variance of VS rates (level factor), 2nd explains an additional 4.4% (slope factor).
  - So two factors explain nearly all the variance of VS rates. This is simpler than the term structure of bond yields. No need for curvature.
  - Egloff et al. (2010), Gatheral (2008) and Amengual (2008) all have two-factor models.

Principal Component Analysis of VS Rates



2.2. Model-Free Jump Component in VS

- We start with the model-free method (see Neuberger (1994), Dupire (1993), Carr and Madan (1998), Demeterfi et al. (1999), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Jiang and Oomen (2008), Carr and Wu (2009) and Carr and Lee (2010).)

- The replication is model-free in the sense that the stock price can follow the general model, but with the restriction $\lambda_t = 0$ and/or $J_t = 0$.

- We wish to quantify the jump component in VS rates.

- If the stock price has a jump component, this replication no longer holds.

  - This observation makes it possible to assess whether VS rates embed a priced jump component.

  - We compare the variance swap rate and the cost of the replicating portfolio.

  - If the difference between the two is zero, then the stock price has no jump component and the VS rate cannot embed a priced jump component.

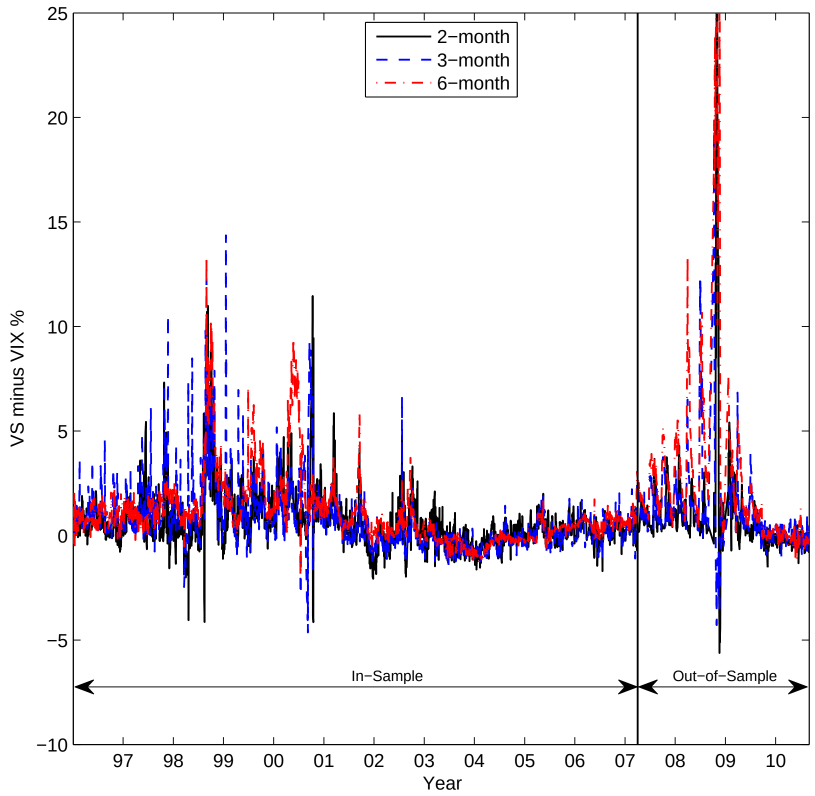
  - If the difference is not zero, a priced jump component is likely to be reflected in such a difference and thus in the VS rate.

$$
\begin{aligned}
\mathsf{VS}_{t,t+\tau} &= E_t^Q[\mathsf{QV}_{t,t+\tau}] \\
&= \frac{2}{\tau}\int_0^\infty \frac{\Theta_t(K,t+\tau)}{K^2}dK \\
&+ \frac{2}{\tau}E_t^Q \sum_{u=N_t}^{N_{t+\tau}} \left[\frac{J_u^2}{2} + J_u + 1 - \exp(J_u)\right] \\
&= \mathsf{VIX}_{t,t+\tau} + \frac{2}{\tau}E_t^Q \sum_{u=N_t}^{N_{t+\tau}} \left[\frac{J_u^2}{2} + J_u + 1 - \exp(J_u)\right]
\end{aligned}
$$

- $\Theta_t(K, t+\tau)$ is the time-t forward price of the out-of-the-money put or call option with strike K and maturity $t+\tau$.
- $\mathsf{VIX}_{t,t+\tau}$ is then the variance swap rate, $\mathsf{VS}_{t,t+\tau}$, when the stock price has no jump component.

- The key point for what follows is that the difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is (up to the discretization error) a model-free measure of the jump component in VS rates, i.e., the last term above.
  - If the jump component is zero, i.e., the jump size $J_u = 0$ and/or the intensity of the counting process N_t is zero, then $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is zero as well.
  - If the jump component is not zero and priced, then $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ tends to be positive.
  - The reason is that the jump term in the square brackets is downward sloping and passing through the origin.
  - If the jump distribution under Q is shifted to the left, suggesting that jump risk is priced, the last expectation tends to be positive.

Time Series of $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ for Different Maturities



- We find that $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ is mostly positive, statistically significant, larger during market turmoils but sizeable also in quiet times.
  - In volatility units, they easily exceed 2% suggesting that they are economically important when compared to an average volatility level of about 20%.
  - A positive difference is not a crisis-only phenomenon, when jumps in stock price are more likely to occur and investors may care more about jump risk.
  - These findings are consistent with the presence of a priced jump component embedded in VS rates.

2.3. A Parametric Stochastic Volatility Model

- The limitations of the data available make it necessary to adopt a parametric structure, with a specification informed by the model-free analysis above, in order to go further.

$$
\begin{aligned}
dS_t/S_{t_-} &= \mu_t dt + (1-\rho^2)^{1/2} v_t^{1/2}\, dW_{1t}^P \\
&\quad + \rho v_t^{1/2}\, dW_{2t}^P + (\exp(J_t^P) - 1)\, dN_t - \nu_t^P\, dt \\
dv_t &= k_v^P (m_t\, k_v^Q / k_v^P - v_t) dt + \sigma_v v_t^{1/2}\, dW_{2t}^P \\
dm_t &= k_m^P (\theta_m^P - m_t) dt + \sigma_m m_t^{1/2}\, dW_{3t}^P
\end{aligned}
$$

where $\mu_t = r - \delta + \gamma_1 (1-\rho^2) v_t + \gamma_2 \rho v_t + (g^P - g^Q)\lambda_t$, r is the risk free rate and δ the dividend yield, both taken to be constant for simplicity only.

- The instantaneous correlation between stock returns and spot variance changes, ρ, captures the leverage effect.

- The random jump size, J_t^P, is independent of the filtration generated by the Brownian motions and jump process, and normally distributed with mean μ_j^P and variance σ_j^2.

- The jump intensity is $\lambda_t = \lambda_0 + \lambda_1 v_t$, where λ_0 and λ_1 are positive constants. This specification allows for more price jumps to occur during more volatile periods, with the intensity bounded away from 0 by λ_0.

- The spot variance, v_t, follows a two-factor model where $m_t \, k_v^Q / k_v^P$ is its stochastic long-run mean or central tendency. The speed of mean reversion is k_v^P under P, k_v^Q under Q and $k_v^P = k_v^Q - \gamma_2\sigma_v$, where γ_2 is the market price of risk for W_{2t}^P;

- The stochastic long run mean of v_t is controlled by m_t which follows its own stochastic mean reverting process and mean reverts to a positive constant θ_m^P, when the speed of mean reversion k_m^P is positive.

- Typically, v_t is fast mean reverting and volatile to capture sudden movements in volatility, while m_t is more persistent and less volatile to capture long term movements in volatility.

- Under Q, the ex-dividend price process evolves as

$$\begin{aligned} dS_t/S_{t_-} &= (r-\delta)\,dt + (1-\rho^2)^{1/2} v_t^{1/2}\,dW_{1t}^Q \\ &\quad + \rho v_t^{1/2}\,dW_{2t}^Q + (\exp(J_t^Q)-1)\,dN_t^Q - \nu_t^Q\,dt \\ dv_t &= k_v^Q(m_t - v_t)dt + \sigma_v v_t^{1/2}\,dW_{2t}^Q \\ dm_t &= k_m^Q(\theta_m^Q - m_t)dt + \sigma_m m_t^{1/2}\,dW_{3t}^Q \end{aligned}$$

2.4. The Term Structure of VS Rates

- Given the stochastic volatility model above, VS rates can be calculated:

$$\begin{aligned}\mathrm{VS}_{t,t+\tau} &= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q[v_s]ds + \frac{1}{\tau}E^Q[J^2]\,E_t^Q[N_{t+\tau}-N_t] \\ &= E^Q[J^2]\lambda_0 + (1+\lambda_1 E^Q[J^2]) \\ &\times [(1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q + \phi_v^Q(\tau)v_t + \phi_m^Q(\tau)m_t]\end{aligned}$$

where $E^Q[J^2] = E_t^Q[J^2]$, as the random jump size is time-homogeneous, and

$$\phi_v^Q(\tau) = \frac{(1-e^{-k_v^Q\tau})}{k_v^Q\tau}$$

$$\phi_m^Q(\tau) = \frac{\left(1+e^{-k_v^Q\tau}k_m^Q/(k_v^Q-k_m^Q) - e^{-k_m^Q\tau}k_v^Q/(k_v^Q-k_m^Q)\right)}{k_m^Q\tau}$$

- This is a parsimonious parametric model consistent with the nonparametric analysis of VS rates.

- It nests the Heston model

  – By imposing the restriction $m_t = \theta_v^Q$ for all t and $\lambda_0 = \lambda_1 = 0$.

  – The VS rate becomes $\mathrm{VS}_{t,t+\tau} = (1 - \phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)v_t$, i.e., a weighted average of v_t and θ_v^Q.

  – Hence, the term structure of VS rates can only be upward or downward sloping at each point in time, depending on whether $v_t < \theta_v^Q$ or $v_t > \theta_v^Q$, respectively.

  – Moreover, the persistence of VS rates is then the same for all maturities as only one factor, v_t, is driving all VS rates.

3. Likelihood-Based Estimation

- The procedure we employ then combines time series information on the S&P500 returns and cross sectional information on the term structures of VS rates in the same spirit as in other derivative pricing contexts

- Hence, P and Q parameters, including risk premia, are estimated jointly by exploiting the internal consistency of the model, thereby making the inference procedure theoretically sound.

- Let $X_t' = [\log(S_t), Y_t']$ denote the state vector, where $Y_t = [v_t, m_t]'$ is latent and will be extracted from actual VS rates.

- Likelihood-based estimation requires evaluation of the likelihood function of index returns and term structures of variance swap rates for each parameter vector during a likelihood search.

- The procedure for evaluating the likelihood function consists of the following steps.

  – First, we extract the unobserved state vector Y_t from a set of benchmark variance swap rates, assumed to be observed without error:

$$\begin{bmatrix} \mathsf{VS}_{t,t+\tau_1} \\ \vdots \\ \mathsf{VS}_{t,t+\tau_\ell} \end{bmatrix} = \begin{bmatrix} a(\tau_1;\Theta) \\ \vdots \\ a(\tau_\ell;\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_1;\Theta)' \\ \vdots \\ b(\tau_\ell;\Theta)' \end{bmatrix} Y_t$$

where Θ denotes the model parameters, where

$$a(\tau;\Theta) = E^Q[J^2]\lambda_0 + (1+\lambda_1 E^Q[J^2])(1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q$$
$$b(\tau;\Theta)' = (1+\lambda_1 E^Q[J^2])\ [\phi_v^Q(\tau), \quad \phi_m^Q(\tau)].$$

– Second, we evaluate the joint likelihood of the stock returns and extracted time series of latent states, using the closed-form approximation to the likelihood function of Aït-Sahalia (2002) for $p_X(x_\Delta|x_0, N_\Delta = j)$ and

$$\begin{aligned} p_X(x_\Delta|x_0) = & \ p_X(x_\Delta|x_0, N_\Delta = 0)\,\mathrm{Pr}(N_\Delta = 0) \\ & + p_X(x_\Delta|x_0, N_\Delta = 1)\,\mathrm{Pr}(N_\Delta = 1) + o(\Delta) \end{aligned}$$

where $\mathrm{Pr}(N_\Delta = j)$ is the probability that j jumps occur during the day of length Δ.

– Third, we multiply this joint likelihood by a Jacobian determinant to compute the likelihood of observed data, namely index returns and term structures of VS rates.

– Finally, for the remaining VS rates assumed to be observed with error, we calculate the likelihood of the observation errors induced by the previously extracted state variables.

– The product of the two likelihoods gives the joint likelihood of the term structures of all variance swap rates and index returns

* the asset prices, A_t, are given by an affine transformation of X_t

$$
\begin{aligned}
A_t = \begin{bmatrix} \log(S_t) \\ \mathrm{VS}_{t,\cdot} \end{bmatrix} &= \begin{bmatrix} \log(S_t) \\ a(\Theta) + b(\Theta)Y_t \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ a(\Theta) \end{bmatrix} + \begin{bmatrix} 1 & 0' \\ 0 & b(\Theta) \end{bmatrix} X_t
\end{aligned}
$$

and rewritten in matrix form reads $A_t = \tilde{a}(\Theta) + \tilde{b}(\Theta)X_t$, with obvious notation.

* the Jacobian term of the transformation from X_t to A_t is therefore

$$\det\left|\frac{\partial A_t}{\partial X_t'}\right| = \det\left|\tilde{b}(\Theta)\right| = \det\left|b(\Theta)\right|.$$

* in the model, $\det|b(\Theta)| = (1+\lambda_1 E^Q[J^2])^2\ (\phi_v^Q(\tau_1)\phi_m^Q(\tau_2) - \phi_v^Q(\tau_2)\phi_m^Q(\tau_1))$.

* Since $X_t = \tilde{b}(\Theta)^{-1}[A_t - \tilde{a}(\Theta)]$,

$$p_A(A_\Delta|A_0;\Theta) = \det\left|b(\Theta)^{-1}\right| \\ p_X(\tilde{b}(\Theta)^{-1}[A_\Delta - \tilde{a}(\Theta)]|\tilde{b}(\Theta)^{-1}[A_0 - \tilde{a}(\Theta)];\Theta).$$

– We then maximize that joint likelihood over the parameter vector.

4. Fitting Variance Swap Rates

4.1. In-Sample Estimation

- We use as in-sample for estimation the period from January 4, 1996 to April 2, 2007.

	Estimate	Std.Err.		Estimate	Std.Err.		Estimate	Std.Err.
κ_v^P	4.803	0.353	ρ	-0.713	0.010	σ_{e_1}	0.004	0.000
σ_v	0.419	0.009	γ_1	-2.545	4.206	σ_{e_2}	0.002	0.000
κ_m^P	0.234	0.086	γ_2	-2.244	0.851	σ_{e_3}	0.003	0.000
θ_m^P	0.043	0.016	γ_3	-0.673	0.610	ρ_e	-0.088	0.006
σ_m	0.141	0.002	μ_j^P	0.010	0.008			
λ_0	3.669	0.621	μ_j^Q	-0.001	0.009			
λ_1	44.770	17.227	σ_j	0.038	0.003	Log-likelihood	74,381.8	

- The spot variance is relatively fast mean-reverting as k_v^P implies a half-life of 36 days.

- Its stochastic long run mean is slowly mean reverting with a half-life of almost 3 years.

- The instantaneous volatility of v_t is almost 3 times that of m_t.

- The correlation between stock returns and variance changes, ρ, is -0.7, confirming the presence of a strong leverage effect.

- The long-run average volatility is 21%

- Both γ_2 and γ_3 are negative, implying negative instantaneous variance risk premia.

- The correlation parameter for the VS pricing errors, ρ_e, is close to 0, suggesting that the model does not produce systematic pricing errors.

- The expected jump size is positive under the objective probability measure, μ_j^P, and slightly negative under the risk-neutral measure, μ_j^Q, suggesting a positive jump risk premium.

- Estimates of jump intensity implies six jumps per year on average.

- We also estimate three nested models:
  - two-factor model with constant jump intensity (setting $\lambda_1 = 0$)
  - two-factor model with no jump component (setting $\lambda_0 = \lambda_1 = 0$)
  - the Heston model (setting $\lambda_0 = \lambda_1 = 0$ and $m_t = \theta_v^P$ for all t).
- The nested models are rejected: each restriction significantly deteriorates the joint fit of VS rates and S&P500 returns.

4.2. Out-of-Sample

- We conduct all subsequent analyses using two subsamples.

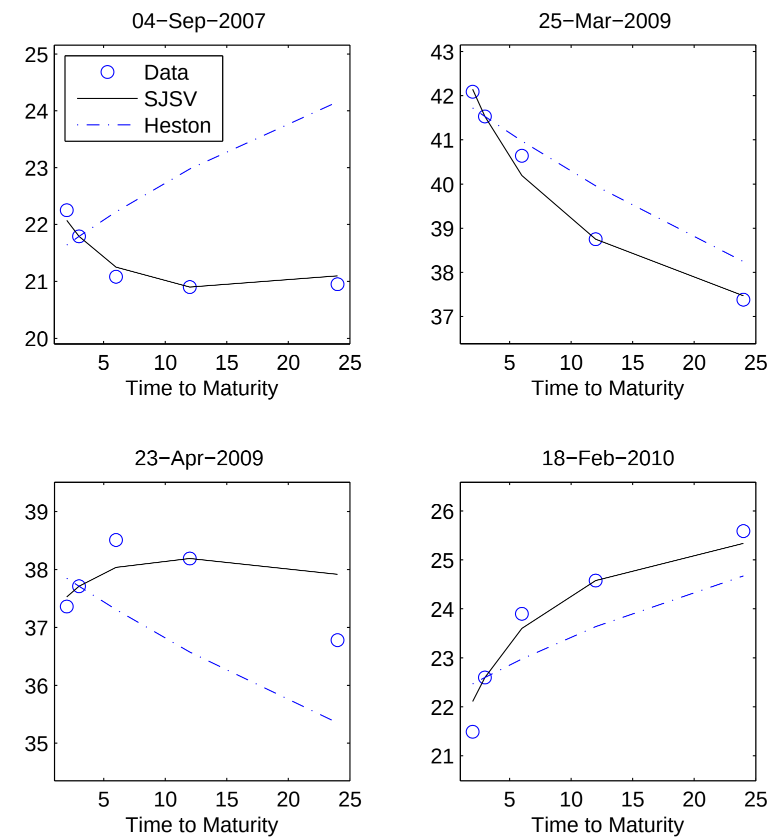
- Data from January 4, 1996 to April 2, 2007 are used for in-sample estimation.

- The remaining sample data, from April 3, 2007 to September 2, 2010, are used for out-of-sample analysis and robustness checks.

- The second subsample includes the financial crisis of Fall 2008 which was not experienced (nor likely anticipated) in the prior fitting sample.

- We compute pricing errors, the model-based VS rate minus the observed VS rate

Out-of-Sample Fit of the Parametric Model



- The model fits VS rates well both in- and out-of-sample and significantly outperforms the Heston model.

- For example, its RMSE is 6 times smaller than that of the Heston model when fitting 24-month to maturity VS rates.

5. Risk Premia: Equity and Volatility

- One advantage of modeling the underlying asset returns jointly with the VS rates is that the resulting model produces estimates of risk premia for both sets of variables, including in particular estimates of the classical equity premium.

- We distinguish between the spot or instantaneous risk premia at each instant t and the integrated ones, defined over each maturity τ.

- In each case, the model provides a natural breakdown between the continuous and jump components of the respective risk premia.

- Results
  - We find an important role of the jump component in the spot equity risk premium.
  - The longer the maturity, the larger the negative risk premium, especially during turbulent times.
  - Market crashes impact and propagate differently throughout the term structure, with the short-end being more affected, but the long-end exhibiting more persistency.

5.1. Spot Risk Premia

- The model contains four instantaneous or spot risk premia: A Diffusive Risk Premium (DRP), a Jump Risk Premium (JRP), a Variance Risk Premium (VRP), and a Long-Run Mean Risk Premium (LRMRP)

$$\text{DRP}_t = (\gamma_1(1-\rho^2) + \gamma_2\rho)v_t, \qquad \text{JRP}_t = (g^P - g^Q)(\lambda_0 + \lambda_1 v_t)$$
$$\text{VRP}_t = \gamma_2\sigma_v v_t, \qquad \text{LRMRP}_t = \gamma_3\sigma_m m_t.$$

- DRP is the remuneration for diffusive-type risk only (due to Brownian motions driving stock prices). JRP is the remuneration for the jump component in stock price.

- The instantaneous Equity Risk Premium (ERP) is the sum of the two, i.e., $\text{ERP}_t = \text{DRP}_t + \text{JRP}_t$.

- The mean growth rates of v_t and m_t are different under the probability measures P and Q, and such differences are given by VRP_t and LRMRP_t, respectively.

- As γ_2 and γ_3 are estimated to be negative, VRP and LRMRP are both negative, and on average v_t and m_t are higher under Q than under P.

- The negative sign of the variance risk premium is not abnormal.
  - The risk premium for return risk is positive, because investors require a higher rate of return as compensation for return risk.
  - On the other hand, investors require a lower level of variance as compensation for variance risk, hence the negative variance risk premium.
  - Risk-averse investors dislike both higher return variance and higher variance of the return variance.

- During our in-sample period, January 1996 to April 2007, the average ERP is 6.8%.
  - Notably, 5.5% is due to the jump risk premium, which thus accounts for a large fraction of the equity risk premium.
  - Jumps in prices are rare events, but jump risk is important as it cannot be hedged easily.
  - The average VRP is also substantial and around −3.4%, while the LRMRP is much lower and around −0.4%.
  - During the out-of-sample period, April 2007 to September 2010, all risk premia nearly doubled reflecting the unprecedented turmoil in financial markets.

5.2. Integrated Risk Premia

- The annualized integrated Equity Risk Premium (IERP) is defined as

$$\text{IERP}_{t,t+\tau} = E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau$$

  and represents the expected excess return from buying and holding the S&P500 index from t to $t+\tau$.

- Extensive research has been devoted to study levels and dynamics of the IERP, in particular investigating the so-called equity premium puzzle.

- Much less attention has been devoted to study the term structure of the IERP.

- The IERP can be decomposed in the continuous and jump part, i.e.,

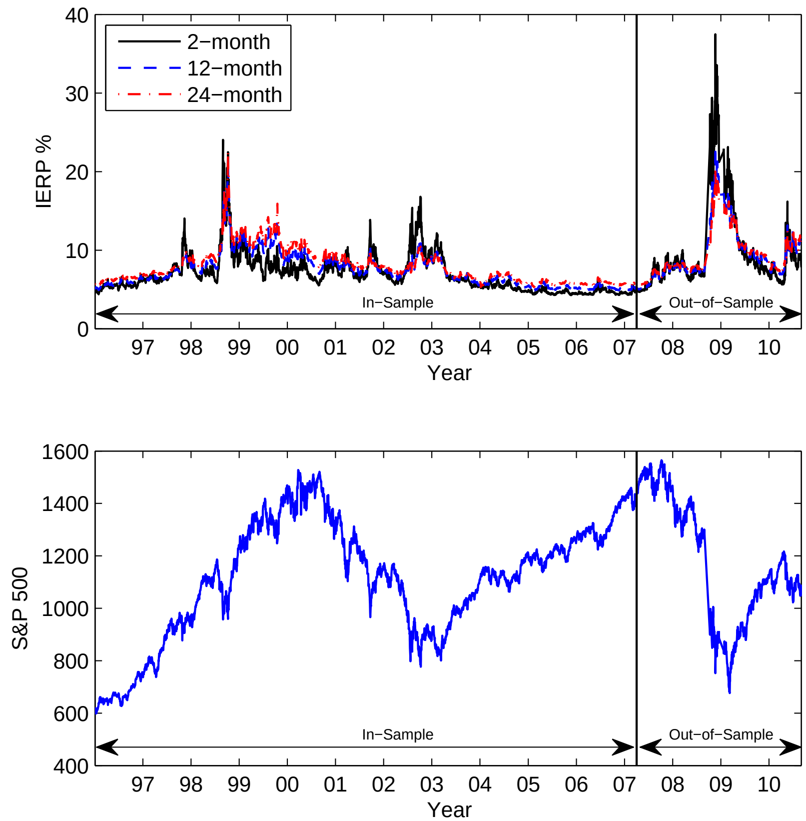
$$\text{IERP}_{t,t+\tau} = \text{IERP}^{c}_{t,t+\tau} + \text{IERP}^{j}_{t,t+\tau}$$

- The analytical expressions are of the form $\exp(A(\tau)+B(\tau)v_t+C(\tau)m_t)$, where $A(\tau)$, $B(\tau)$ and $C(\tau)$ are positive functions when evaluated at the model's parameter estimates.

- Therefore, in quiet times, when the spot variance v_t and its stochastic long run mean m_t are low, IERPs are low as well.

- When asset prices fall and v_t and/or m_t increase, IERPs increase as well, reflecting distressed asset prices. Thus, the IERP is countercyclical.

Time Series of Estimated IERP



- The entire term structure of the IERP exhibits significant variation over time, with the short-end being more volatile than the long-end.

- When the S&P500 steadily increased during 2005–07, the 2-month IERP dropped at the lowest level, around 4.5%, during our sample period.

- The term structure was upward sloping with the 24-month IERP at 6%.

- At the end of 2008 and beginning of 2009, the term structure of the IERP became downward sloping with the 2-month IERP reaching its highest value in decades.

- On November 20, 2008, the 2-month IERP was as high as 37%, and between October and December 2008, the IERP was above 25% on various occasions, mirroring the fall of the index.

- Indeed, from mid-September to mid-November 2008, the S&P500 index dropped from 1,200 to 750, loosing 37% of its value.

- On March 9, 2009, it reached its lowest value in more than a decade, at 677, and then recovered 35% of its value within the next two months. Such large swings in the S&P500 index suggest that the large model-based estimates of the IERP are reasonable.

- Integrated Variance Risk Premium
  - The annualized integrated variance risk premium (IVRP) is defined as $\text{IVRP}_{t,t+\tau} = E_t^P[\text{QV}_{t,t+\tau}] - E_t^Q[\text{QV}_{t,t+\tau}]$ and represents the expected profit to the long side of a VS contract, which is entered at time t and held till maturity $t + \tau$.
  - Investors regard volatility increases as unfavorable events and are willing to pay large premia, i.e., large VS rates, to insure against such volatility increases.
  - The term structure of IVRP is on average downward sloping.

- We study the impact of negative jumps and the induced term structure of variance risk premia.

  - As many investors are "long in the market" and the leverage effect is very pronounced, negative jump prices are perceived as unfavorable events and thus can carry particular risk premia. The contribution of negative jumps to the IVRP is given by

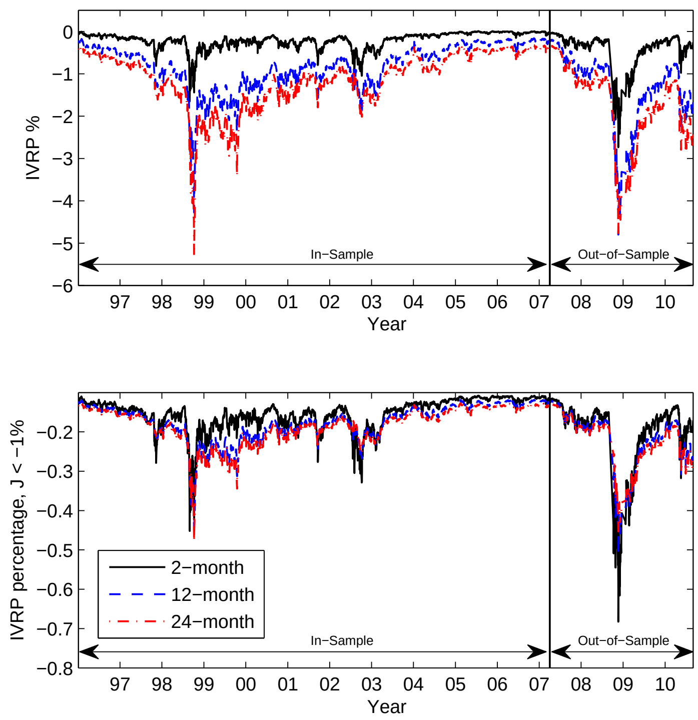
    $$\mathsf{IVRP}(k)^{j}_{t,t+\tau} = E^P_t[\mathsf{QV}^{j}_{t,t+\tau}\,1_{\{J<k\}}] - E^Q_t[\mathsf{QV}^{j}_{t,t+\tau}\,1_{\{J<k\}}]$$

    where $1_{\{J<k\}}$ is the indicator function of the event $J < k$.

  - We set $k = -1\%$, i.e., we study the contribution of daily jumps below -1% to the IVRP.

  - Between April 2007 and September 2010, the out-of-sample period, $\mathsf{IVRP}(k)^{j}_{t,t+\tau}$ accounts for nearly half of the IVRP at the 2-month horizon.

Time Series of Estimated IVRP



- Similarly to the IVRP, the term structure of IVRP$(k)^j_{t,t+\tau}$ is generally downward sloping in quiet times.

- However, in contrast to IVRP, during market crashes the term structure of IVRP$(k)^j_{t,t+\tau}$ becomes suddenly upward sloping, reflecting the large jump risk due to a price fall.

- In Fall 2008, the two-month IVRP$(k)^j_{t,t+\tau}$ exhibited the largest negative drop and took several months to revert to more normal values.

- This is consistent with investors' willingness to ensure against a market crash increasing after a price fall.

6. The Expectation Hypothesis

- Several studies have investigated whether term structures of interest rates, exchange rates or option implied volatilities can predict future bond yields, spot exchange rates or underlying asset volatilities, respectively.

- The traditional approach is to regress the variable of interest y_t (e.g., bond yield at time t) on a constant and its predictor x_t (e.g., forward rate at time $t-1$ with maturity t).

- In the regression $y_t = \alpha + \beta\, x_t + \varepsilon_t$, the hypothesis of interest is whether the predictor is unbiased, i.e., $\alpha = 0$, and/or efficient, i.e., $\beta = 1$, in which case, the so-called Expectation Hypothesis (EH) holds.

- In our setting, a natural question is whether the VS rate provides an unbiased and/or efficient prediction of future realized variance.

- The standard testing procedure would be to run a time series regression of future, ex-post realized variance, $\text{QV}_{t,t+\tau}$, on a constant and the VS rate, $\text{VS}_{t,t+\tau}$.
  - An alternative interpretation of the EH is in terms of risk premia. By definition of IVRP, $\text{QV}_{t,t+\tau} = \text{VS}_{t,t+\tau} + \text{IVRP}_{t,t+\tau} + \varepsilon_{t,t+\tau}$, where the last, zero mean, error term disappears when taking the time-t conditional P-expectation.
  - This equation shows that if $\text{IVRP}_{t,t+\tau}$ is zero (or is stochastic but with zero mean and uncorrelated with $\text{VS}_{t,t+\tau}$), the regression above would produce an intercept of zero and slope of one, i.e., EH would hold.
  - If $\text{IVRP}_{t,t+\tau}$ is constant (or is stochastic but with constant mean and uncorrelated with $\text{VS}_{t,t+\tau}$), the intercept would equal this constant value and the slope would still be one.

- If, indeed, $\text{IVRP}_{t,t+\tau}$ and $\text{VS}_{t,t+\tau}$ are driven by some common factors (i.e., the two are correlated), when regressing the future realized variance on the VS rate only, an important variable is missing in the regression. $\text{IVRP}_{t,t+\tau}$ would be absorbed in the regression error term, which would be correlated with the VS rate regressor, producing inconsistent estimates.
- Thus, the intercept will not be zero and the slope will not be one. In other words, a rejection of the EH would imply a non-trivial dynamic of the IVRP.

- Using the model, we can calculate the theoretical values of α and β:

$$\beta = \frac{\mathrm{Cov}^P[\mathrm{VS}_{t,t+\tau}, \frac{1}{\tau}\int_t^{t+\tau} v_s\, ds]}{\mathrm{Var}^P[\mathrm{VS}_{t,t+\tau}]} = \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)}$$

$$\alpha = E^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds] - \beta E^P[\mathrm{VS}_{t,t+\tau}]$$

$$= \theta_v^P - \frac{\phi_v^P(\tau)}{\phi_v^Q(\tau)}((1 - \phi_v^Q(\tau))\theta_v^Q + \phi_v^Q(\tau)\theta_v^P)$$

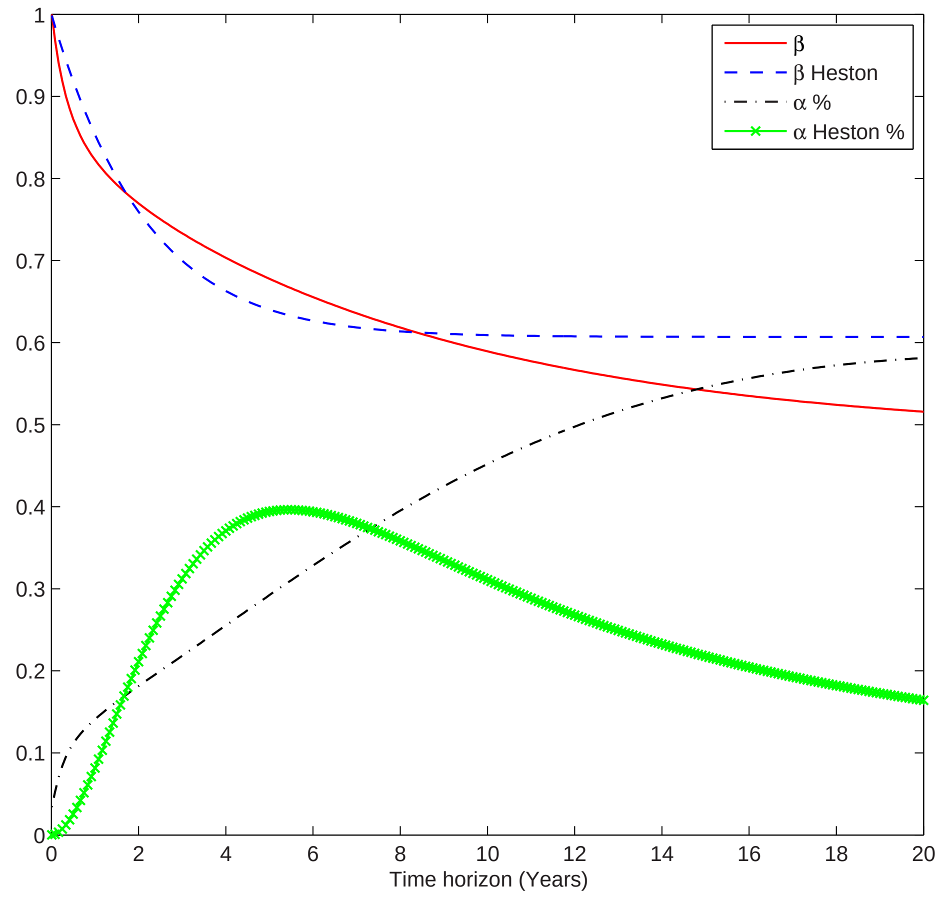
$$R^2 = \frac{\beta^2 \mathrm{Var}^P[\mathrm{VS}_{t,t+\tau}]}{\mathrm{Var}^P[\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds]} = \phi_v^P(\tau)^2 / \left(\frac{2\,(e^{-k_v^P \tau} - 1 + k_v^P \tau)}{\tau^2\,(k_v^P)^2} \right)$$

- When $\tau \longrightarrow 0$, $\beta(\tau) \longrightarrow 1$ and $\alpha(\tau) \longrightarrow 0$. Thus, for very short maturities VS rates are efficient and unbiased predictors of future realized variances. This result is essentially due to the annualized realized variance, $\frac{1}{\tau}\int_t^{t+\tau} v_s\, ds$, converging to the spot variance v_t, when $\tau \longrightarrow 0$.

- When $\tau \longrightarrow +\infty$, $\beta(\tau) \longrightarrow k_v^Q / k_v^P$ and $\alpha(\tau) \longrightarrow 0$. Hence, for very long maturities VS rates are inefficient but unbiased predictor of future realized variance. As $k_v^Q = k_v^P + \gamma_2 \sigma_v$ and γ_2 is estimated to be negative, $k_v^Q / k_v^P < 1$.

- This suggests that VS rates tend to overestimate future RV.

Estimated $\alpha(\tau)$ and $\beta(\tau)$



- For the range of VS rate maturities in our sample (up to 2 years), $\beta(\tau)$ decreases.

- Hence, predictions of future realized variances based on VS rates appear to deteriorate quickly.

- When $\tau = 2$ years, $R^2(\tau)$ is about 70% in the general model.

7. Shorting Variance Swaps?

- As the ex-ante IVRP is negative, shorting VS generates a positive return on average.

- Moreover, in contrast to stock returns, volatility is partially predictable because it follows a persistent and mean reverting process.

  - For example, if today's volatility is high, it is likely that it will remain high in the near future, but eventually will revert to lower levels.

  - In that case, a sensible investment may be to go short in a long-term variance swap contract or "sell volatility," speculating that long-term future volatility will be lower than the current variance swap rate.

- In order to examine the extent to which the large variance risk premia potentially translate into economic gains, we consider a simple but relatively robust trading strategy involving VS.
  - The strategy is to short the VS contract only when the expected profit is large enough and precisely n times larger than its ex-ante, model-based standard deviation.
  - When $n = 0$, the VS contract is shorted as soon as the expected profit is positive. When $n > 0$, the contract is shorted less often.

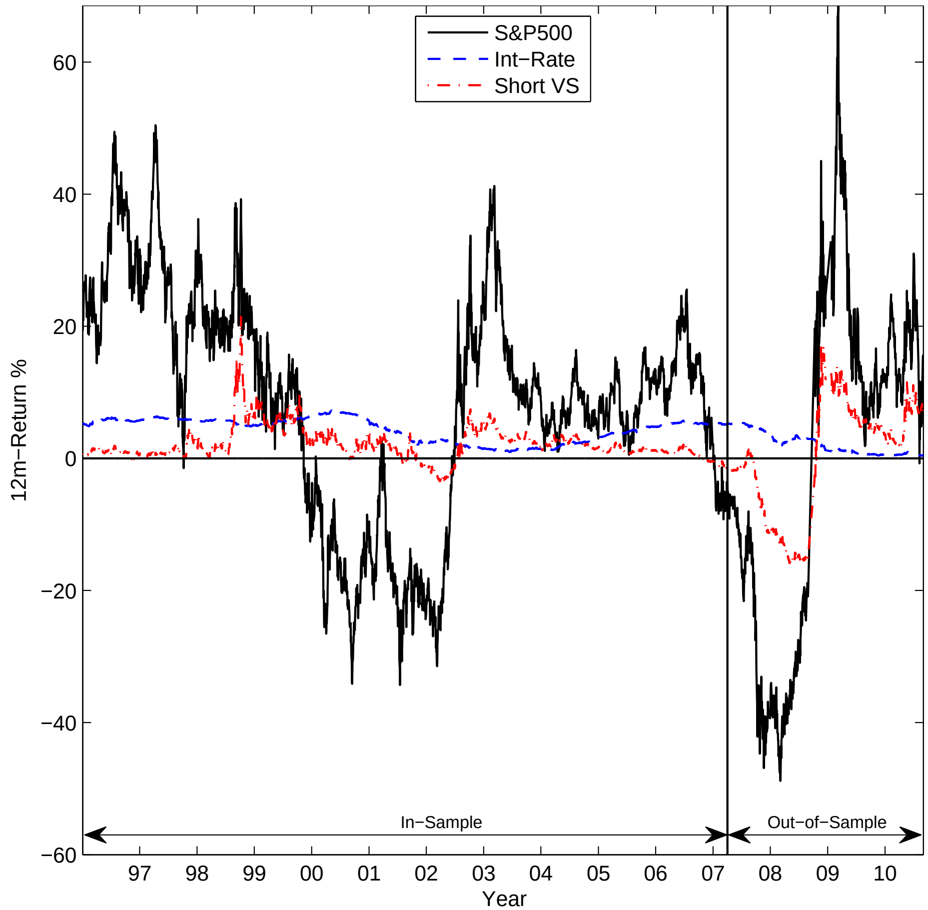
- As a benchmark, we consider the following trading strategy based on the S&P500 index.
  - If at day t the VS contract with maturity $t+\tau$ is shorted, we invest \$1 in the S&P500 index and liquidate the position at day $t+\tau$.
  - The actual return is computed using S&P500 index prices. This buy-and-hold strategy is repeated for each day t in our sample.

- Shorting VS appears to be significantly more profitable than investing in the S&P500 index, over the same time horizons.

- Sharpe ratios from investing in VS are nearly uniformly and significantly increasing in the threshold n.

- The only exception is from shorting the 2-month VS contract in the out-of-sample period, as such positions suffered large losses during the Fall of 2008.

Estimated Returns of the Strategy



	In-Sample Sharpe Ratios							
	Short Variance Swap				Long S&P500			
Horizon	2	3	6	12	2	3	6	12
Threshold								
Always	0.56	0.59	0.67	0.84	0.13	0.15	0.21	0.24
0	0.55	0.59	0.67	0.84	0.08	0.15	0.21	0.24
1	0.60	0.71	1.15	0.94	0.12	0.13	0.50	0.06
2	0.78	1.35	1.91	1.75	0.53	0.77	1.12	0.21
3	0.98	1.60	2.14	2.71	0.72	1.47	1.59	1.17
	Out-of-Sample Sharpe Ratios							
	Short Variance Swap				Long S&P500			
Always	0.20	0.15	0.08	0.02	0.01	−0.06	−0.05	−0.11
0	0.47	0.15	0.08	0.02	0.13	−0.06	−0.05	−0.11
1	0.45	0.19	0.10	0.05	0.41	0.02	0.02	−0.04
2	0.20	1.13	1.15	1.45	0.21	0.53	1.01	1.02
3	−0.10	0.91	1.72	2.79	−0.14	0.28	1.19	1.97

- The losses during 2008 reflect jump and volatility risk that short positions are carrying, but they are smaller than the losses from the buy-and-hold S&P500 strategy.

- Long positions in the S&P500 generate substantial more volatile returns.

- Interestingly, shorting VS does not appear to suffer from the "picking up nickels in front of steamroller" syndrome during the period we looked at, despite the inclusion out-of-sample of the 2008–2009 financial crisis.

- Shorting VS provides some diversification benefits as the correlations show:

	In-Sample			Out-of-Sample		
	S&P500	Int-Rate	Short VS	S&P500	Int-Rate	Short VS
	2-month Returns			2-month Returns		
S&P500	1.00	0.09	0.57	1.00	−0.31	0.63
Int-Rate	0.09	1.00	−0.01	−0.31	1.00	−0.25
Short VS	0.57	−0.01	1.00	0.63	−0.25	1.00
	12-month Returns			12-month Returns		
S&P500	1.00	0.05	0.30	1.00	−0.64	0.91
Int-Rate	0.05	1.00	−0.01	−0.64	1.00	−0.54
Short VS	0.30	−0.01	1.00	0.91	−0.54	1.00

8. Conclusions

- Significant and time-varying jump risk component in VS rates.
- The term structure of variance risk premium is negative and generally downward sloping.
- Going short long-term VS contracts is more profitable on average than going short short-term contracts.
- Variance risk premia due to negative jumps exhibit similar features in quiet times but have an upward sloping term structure in turbulent times.
- Short-term variance risk premia mainly reflect investors' "fear of crash," rather than the impact of stochastic volatility on the investment opportunity set.